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ACOSS NINE (ACTIVE CONTROL OF SPACE STRUCTURES)

Control Dynamics Company

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ACOSS NINE (ACTIVE CONTROL OF SPACE STRUCTURES)

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ACOSS NINE (ACTIVE CONTROL OF SPACE STRUCTURES)

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Contractor: Control Dynamics Company
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This is the Control Dynamics Final Report on the analysis and evaluation of Large Space Structures. The research was primarily aimed at mathematical model development and control system analysis and design tool development.

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The Program Manager and Principal Investigator is Dr. Sherman M. Seltzer, and the Co-Investigator is Dr. H. Eugene Worley. With the exception of Sections 4.0 and 5.2, the report was written by them. Professor Bernard A. Asner, Jr. (of the University of Dallas) wrote Section 5.2, and Professor Dragoslav D. Siljak (of the University of Santa Clara) wrote Section 4.0. The contributions of these two consultants to the Control Dynamics Company are gratefully acknowledged.

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1.1 OBJECTIVE

To study active structure control technology as applied to Large Space Structures (LSS). In particular, to develop analytical tools for analyzing their behavior and to develop a means of ensuring and testing for stability. This objective is in support of the overall ACOSS objective which is to develop and understand a generic, unified, structural dynamics and control technology base for LSS with stringent Line-of-Sight (LOS) and figure performance requirements that must be maintained in the presence of on-board and natural disburbances.

1.2 SCOPE

This effort will be restricted to theory development, demonstration and documentation of structural dynamics and control technology for large precision spacecraft. The contractor planned the overall effort, defined the problem, and developed and evaluated solutions. The development and evaluation of solutions involved analytical studies, modeling and simulations. The contractor identified and documented all the analyses, studies, designs, models, simulations, tradeoffs, issues, software, requirements, specifications and data produced under the contract. Important products of the effort are the demonstration and documentation of new control technology.

1.3 STATEMENT OF WORK

The following is a summary of the tasks and technical requirements that comprise the Statement of Work (SOW). The numbers are identical to those used in the SOW.

"4.1. The contractor shall evaluate the existing LSS mathematical models that have been developed by Charles Starke Draper Laboratory (CSDL). The contractor shall accomplish the following tasks:

4.1.1. Development and Integration of Active Control Efforts.

4.1.1.1. The contractor shall synthesize a high order mathematical model of a Large Space System. The model shall be amenable to simplification (for purposes of analytical tractability) by truncation.

4.1.1.2. The contractor shall investigate sampling phenomena arising from a digital version of an on-board controller, using a truncation version of the model developed under part 4.1.1.1. above.

4.1.1.3 The contractor shall integrate the technical approaches of 4.1.1, 4.1.2, and 4.1.3.

4.1.2 Stability of Large Scale Systems.

Using the high order mathematical model of a Large Space System developed under 4.1.1, the contractor shall formulate a suitable mathematical framework. Using the decomposition principle, the contractor shall investigate its applicability.

4.1.3 Development of Parameter Space Technique.

The contractor shall develop the Parameter Space technique to provide a desired set of rigorous stability criteria applicable to the high order model of the Large Space System developed under 4.1.1. Its application to the determination of robustness will be investigated.

4.2. Oral Presentations. Oral presentations shall be given at such times and places as designated by the Contracting Officer."

1.4 SCHEDULE

The original scheduled duration of the contract was the period 1 April 1980 through 31 January 1981. This was subsequently modified to extend the period to 1 April 1981. The reason was to permit coverage of additional ACOSS activities pending the start of a new ACOSS contract.

1.5 ADMINISTRATIVE

There were four modifications to the original contract.

Modification #1. This was to alter the originally estimated man-hours to comply with more accurate subsequent estimates.

Modification #2. This was to change the company address.

Modification #3. This extended the duration of the contract to 28 February 1981.

Modification #4. This extended the duration of the contract to 1 April 1981.

1.6 CONTRACT IMPLEMENTATION

Dr. Seltzer and Dr. Worley evaluated the Large Space Structure mathematical models.

Dr. Worley synthesized the "high order" mathematical model of a Large Space Structure.

Dr. Seltzer investigated sampling phenomena under the development of extensions of the Parameter Space technique.

Professor Siljak and Dr. Seltzer investigated stability aspects of Large Space Structures.

Professor Asner, Dr. Seltzer, and Dr. Worley developed extensions of the Parameter Space technique.

2.0 EVALUATION OF DARPA MODEL

2.1 PRELUDE

The evaluation of any model requires an understanding of the requirements that caused the model to be generated in the first place. The CSDL model was developed with the stated objective to assess the performance, sensitivity, and hardware requirements of the various active structural control methods which are being developed. In addition, the design of the model was driven by the desire to incorporate certain attributes into the overall system characteristics, such as, (1) a structural design based on realistic size and weight, (2) a simple unclassified optical system with associated performance measures and tolerances, and (3) a set of disturbances typical of equipment vibration and attitude control. The resulting model was presented in Report RADC-TR-80-377, "Active Control of Space Structures Interim Report" dated September 1980 by The Charles Stark Draper Laboratory, Inc. The purpose of this section is to review in a cursory manner this model to determine the degree to which the stated objectives are reached.

2.1.1 KINETIC ENERGY

The development of a structural model, or any dynamic model for that matter, can begin with the representation of the kinetic and potential energy of the system under question. The general form of the kinetic energy is presented and then simplified under

the assumption that the kinetic energy is not a function of the generalized coordinates, but rather only a function of the rate of change of the generalized coordinates.

$$T = \tfrac{1}{2} \int \dot{\underline{R}} \cdot \dot{\underline{R}} \, dm$$

$R \equiv$ POSITION OF ANY POINT

$T \equiv$ KINETIC ENERGY

$V \equiv$ POTENTIAL ENERGY

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial v}{\partial \dot{q}_i} = Qi + Q\lambda i$$

$qi \equiv$ GENERALIZED COORDINATE

$Q \equiv$ GENERALIZED FORCE(S)

$Q\lambda i \equiv$ CONSTRAINT FORCE(S)

IF $T = T(qi)$

THEN

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q} i}\right) = Qi + Q\lambda i$$

2.1.2 PLANAR EXAMPLE

We now move from this general form of the kinetic energy, to a planar representation of a structure so that we may be able to more completely understand the physics of the situation. For

this purpose, consider the specification of the motion of a structure as depicted. The kinetic energy and the required partial derivatives can be represented as given.

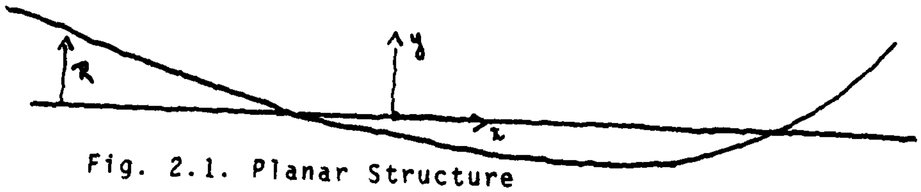


Fig. 2.1. Planar Structure

$$\text{LET } R = Y + X\theta + \sum_{i=1}^{M} \phi_i \; \eta_i$$

ϕ_i = MODE SHAPE

η_i = MODAL GENERALIZED COORDINATE

OR

$$R = \sum_{j=1}^{N} \phi_j \eta_j$$

$$\text{WHERE} \quad \phi_1 \equiv 1 \quad , \quad \eta_1 = Y$$

$$\phi_2 \equiv X \qquad \eta_2 = \theta$$

$$\dot{R} = \sum_{j=1}^{n} \phi_j \dot{\eta}_j \quad ; \quad \frac{\partial \dot{R}}{\partial \dot{\eta}_k} = \phi_k$$

AND

$$\frac{\partial T}{\partial \dot{\eta}_j} = \int \left(\frac{\partial \dot{R}}{\partial \dot{\eta}_k} \right) \cdot \dot{R} \quad dm$$

$$\frac{\partial T}{\partial \dot{\eta}_j} = \int \sum_{i=1}^{N} \phi_j \; \phi_i \; \dot{\eta}_j \; dm$$

IF $\int \phi_j \phi_i \, dm = M_i \, \delta_{ij}$

$$\frac{\partial T}{\partial \dot{\eta}_k} = M_j \, \dot{\eta}_j$$

SIMILARLY WITH POTENTIAL ENERGY

$$\frac{\partial V}{\partial \eta_j} = M_j \, \omega_j^{\,2} \eta_j$$

$\omega_j \equiv$ jth EIGEN VALUE

THUS

$$\boxed{M_j \, \ddot{\eta}_j + M_j \, \omega_j^2 \, \eta_j = Q_j + Q\lambda_j}$$

OR ADDING "MODAL" DAMPING

$$\boxed{M_j \left[\ddot{\eta}_j + 2\xi_j \, \omega_j \, \dot{\eta}_j + \omega_j^2 \, \eta_j\right] = Q_j + Q\lambda_j}$$

ALSO $Q_j \equiv F\phi_j(T) = F\alpha_j$

$\alpha_j \equiv$ SLOPE AT F

2.1.3 GOVERNING DIFFERENTIAL EQUATIONS

We now must call on orthogonality of the "modes" to simplify the equations. Nonorthogonal modes can be used and are used in some situations, but there is increased coupling in the equations with corresponding increased numerical difficulties. The differential equations of motion then follow as given. It is at this point that the concept of equivalent viscous structural damping is introducted into the equations. This representation of energy loss in structures is supported by experimental data when the damping is small (less then 2%), but can be seriously questioned when used for large space structures.

2.1.4 DIFFERENTIAL EQUATIONS WITH CLOSED-LOOP CONTROL

Now it is possible to add the control forces and moments through the concept of a generalized force. We have taken here a standard PD (position and derivative-of-position) controller where the control force or moment is taken to be a linear function of the position and the rate of the body. Of course this information must be sensed, and these control errors are really only an estimate of the actual position and rate of the body in question.

It is now possible to see the influence of the control force and/or moment on the damping and stiffness of any given structural generalized coordinate.

$$\ddot{\eta}_j + 2\xi_j \, \omega_j \, \dot{\eta}_j + \omega_j^2 \, \eta_j = \frac{\partial_j}{M_j}$$

$$\text{NOW LET} \quad F = a_0 \, \theta_1 + a_1 \, \dot{\theta}_1$$

$$\text{AND} \quad \theta = \sum_{i=1}^{N} \theta_i \, \eta_i \quad ; \quad \dot{\theta} = \sum_{i=1}^{N} \theta_i \, \dot{\eta}_i$$

$$F = a_0 \sum_{i=1}^{N} \partial_i \, \eta_i + a_1 \sum_{i=1}^{N} \theta_i \, \dot{\eta}_i$$

OR

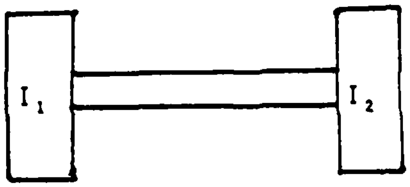
$$\ddot{\eta}_j + \left[2\xi_j \, \omega_j + a_1 \frac{\theta_j \, \alpha_j}{M_j} \right] \dot{\eta}_j + \left[\omega_j^2 + a_0 \frac{\theta_j \alpha_j}{M_j} \right] \eta_j$$

$$= \sum_{\substack{j=1 \\ i \neq j}}^{N} \theta_j \left[a_0 \, \eta_i + a_1 \, \dot{\eta}_i \right]$$

2.1.5 SIMPLE MODEL

It is now beneficial to simplify the model even further and consider a two-mass model as given in the illustration. The equations of motion are simple to derive and therefore are not presented. The resulting characteristic equation is presented in a normalized form. The significant variables remaining in the characteristic equation are the ratio of the inertias of the two bodies, r, and the ratio of the coupling of information

between the two bodies in the error signal, γ. We now will illustrate the effect of these two parameters through the use of root-locus. The numerical results are contained in graphs found at the end of this section.



$$\Phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \Phi_2 = \begin{Bmatrix} {}^1/I_1 \\ -{}^1/I_2 \end{Bmatrix}$$

$$\int \phi_1 \, \phi_2 \, dm \equiv \sum_{j=1}^{N} \phi_{1j} \, \phi_{2_j}^T \, I_j = \frac{I_1}{I_1} - \frac{I_2}{I_2} = 0$$

DEFINE

$r \equiv I_1/I_2$

$\gamma \equiv$ RATIO OF BODY 2 TO BODY 1 INFORMATION CONTAINED IN ERROR SIGNAL

$S \equiv s/\omega_N$

CHARACTERISTIC EQUATION

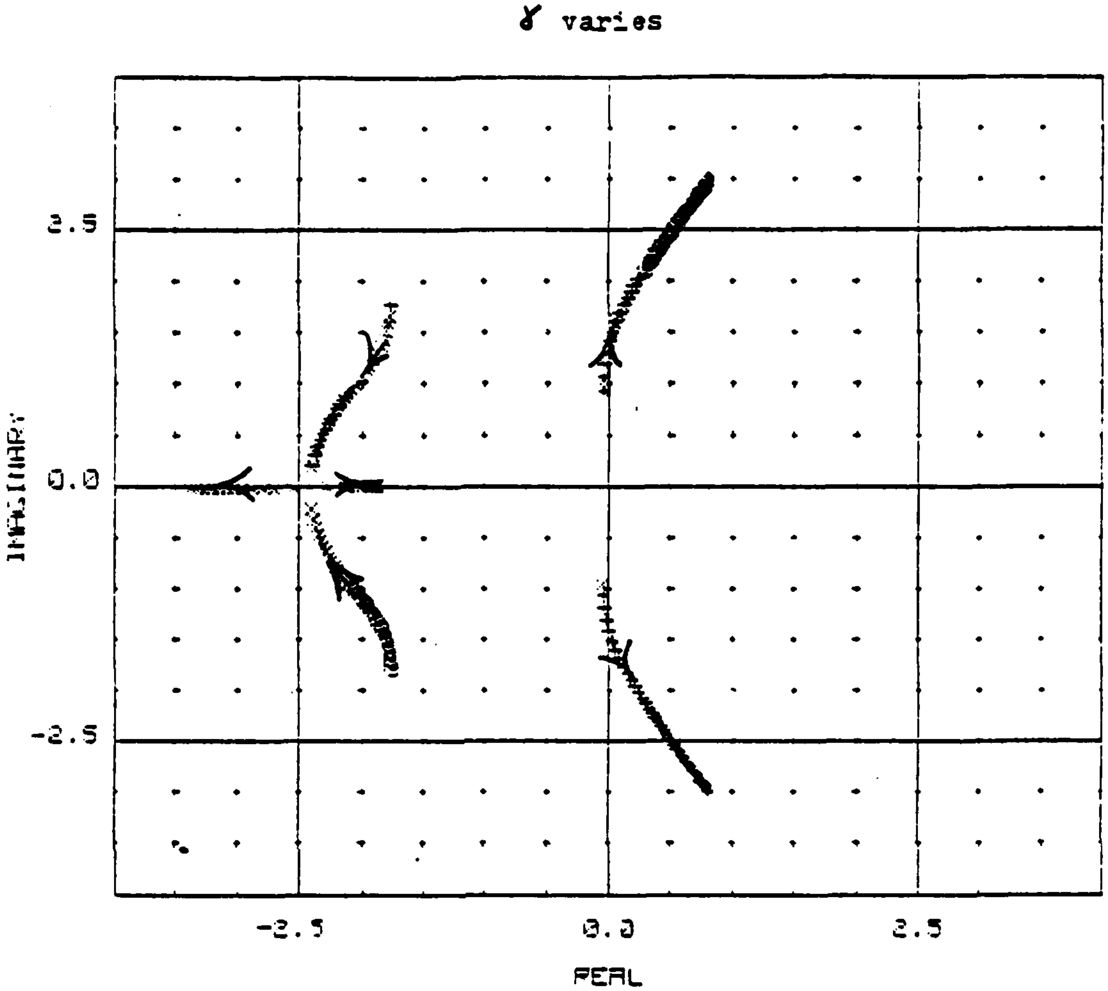
$$S^4 + a_1 S^3 + \left[(1+r) + a_0\right] S^2 + \left[r(1+\gamma)a_1\right] S + ra_0 \, (1+\gamma) = 0$$

Figure 2.3 presents the root locus for the case when the two bodies in our model have equal moments of inertia. (Read here that the modal gains at each of the two points on the structure have equal magnitudes and opposite signs.) The effect of increasing the coupling between the two bodies through the control loop (increasing) is to decrease the damping in the lightly damped mode and to eventually drive it unstable.

Figure 2.4 is a root locus for the case when the control actuator is on a body that has ten times the inertia of the other body. Here it can be seen that the effect of the coupling in the control signal is to drive the system more quickly unstable. The modal deflection at the second point is the culprit, being inversely proportional to the moment of inertia ratio.

2.1.6. OBSERVATIONS

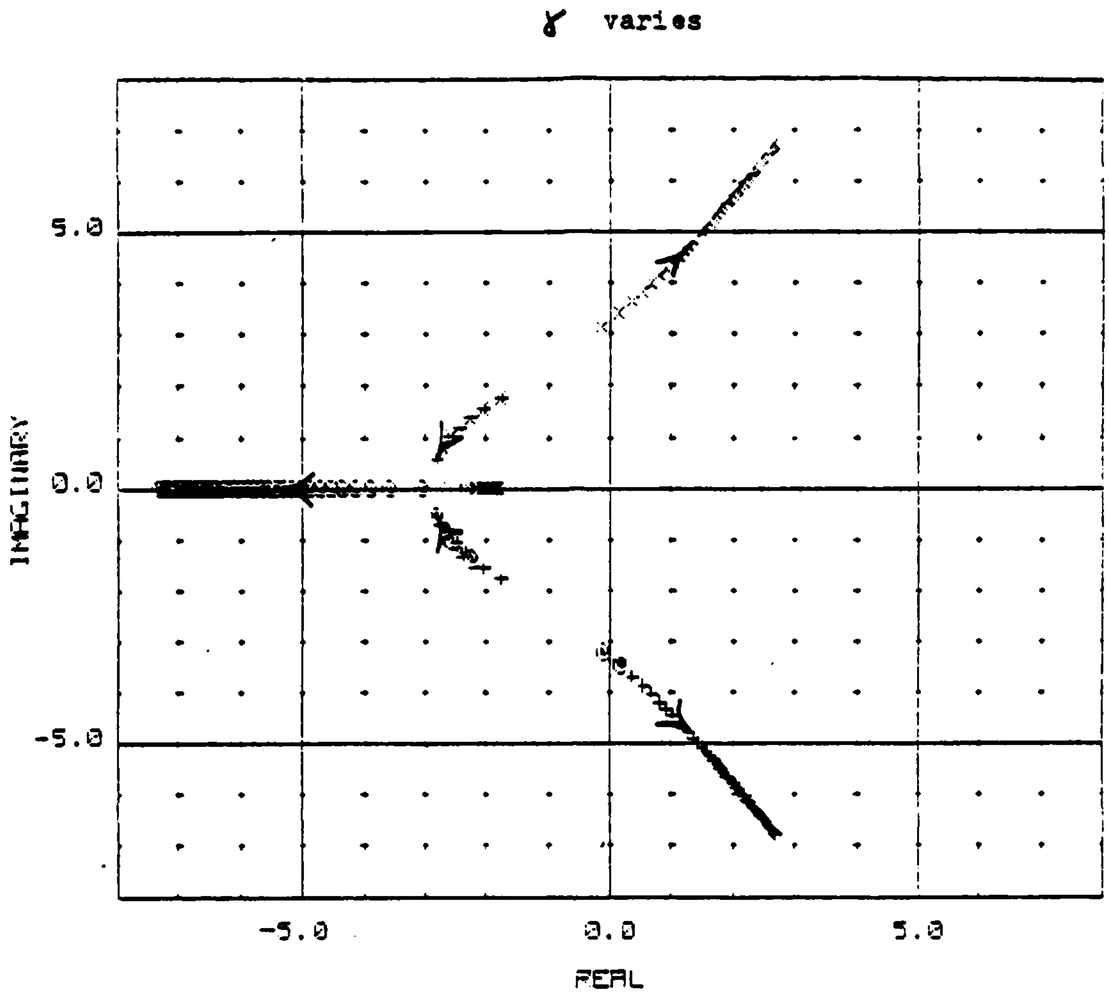
This short journey into the world of flexible body analysis and control was for the purpose of developing an understanding of the manner in which physical systems work together and how control systems play in the game. We believe that it is very important for one to understand the system on which he is working and not just take the most sophisticated computer program that can be found and start turning the crank. There are a lot of pitfalls that can be avoided if one takes a lantern along with him.



CASE #1=+ #2=x #3=0 #4=+

$r = 1.0$; $\zeta = 0.707$; $\omega_n = 2.5$ rad/sec

Fig. 2.3. Root Locus: Equal Masses in Model.



CASE #1=+ #2=x #3=0 #4=+

$r = 10.0$; $\zeta = 0.707$; $w_n = 2.5$ rad/sec with $w_b = 1.0$ rad/sec

Fig. 2.4. Root Locus: Unequal Masses in Model.

2.2 SUMMARY EVALUATION OF CSDL - DEVELOPED MODEL

This model appears to have a good representation of the optical bench. The complex paths that the light rays follow and the lengths of these paths are sure to cause problems if one uses the focal plane error signal in the control loop. However, the subsystem structure is poorly, if not unsatisfactorily, represented. In addition, the design of the isolator is in some question. The design appears to be based on the desire to isolate longitudinal vibrations, while lateral and bending vibrations would have a more harmful effect on the image.

The following equations illustrate one type of simple analysis that can and should be used to determine the efficacy of any model. Here simple mathematical expressions are utilized to yield the result that the CSDL Instrument Section has zero dimension in the z direction. This is a serious flaw in our opinion in that this is the section in which the control actuators are present and also the system rate gyros. We have already illustrated that flexibility between these two devices is a serious problem for control system designers.

INCORRECT MOI

$$I_{xx} = I_{yy} = 2.1 \times 10^3 \ \text{KG-M}^2$$

$$I_{zz} = 4.2 \times 10^3 \ \text{KG-M}^2$$

$$I_{xx} = I_{yy}$$

$$\int (y^2 + z^2)dm = \int (x^2 + z^2)dm$$

$$\int y^2 dm = x^2 dm$$

$$I_{zz} = 2 I_{xx}$$

$$\int (y^2 + x^2) dm = 2 \int (y^2 + z^2) dm$$

$$2 \int y^2 dm = 2 \int (y^2 + z^2) dm$$

$$\underline{\underline{\int z^2 dm = 0}}$$

$\therefore$ <u>ALL MASS IS IN THE xy PLANE</u>

3.0 CONTROL DYNAMICS MODEL

3.1 ANALYSIS

The purpose of this section is to present an alternate approach to the model development. We presented a model to DARPA May 8, 1980 in which we felt were all the ingredients to properly cause a control system designer to work for his lunch. This model was derived using the transfer matrix technique based on an outline presented in "Analytical Methods in Vibrations" by Leonard Meirovitch. Press of time and space will not allow a complete description of the development procedure, but the complete modal model was developed using a Hewlett Packard desktop computer, Model 9835A.

A picture of the model is presented on Fig. 3.1. Here we have attempted to include all the features that would cause the control system designer to have troubles in coming up with a stable and well-performing system. The mode shapes arising out of this model are presented in the following pages.

The first bending mode (Fig. 3.2.) is being driven by the stiffness of the interface and the transverse moment of inertia of the optical bench and the equipment section. For this model, the second mode (Fig. 3.3.) is primarily a torsional mode with the natural frequency being driven by the torsional stiffness of the interface and the roll moment of inertia of the two sections. The third mode (Fig. 3.4.) is a combined bending and torsion mode of the optical bench. The mode shape and frequency of this mode is

determined by the properties of the optical bench alone . The fourth mode (Fig. 3.5.) is a pure bending (first mode) of the optical bench and as such is completely determined by the physical properties of that section. The fifth mode (Fig. 3.6.) is a complex mode involving first bending of the optical bench with flexing of the interface structure. It would be difficult to characterize data to change this mode, but one would expect stiffness of the interface to have some effect. The sixth mode (Fig. 3.7.) represents a second bending of the optical bench and as such is a brother of mode #4. The seventh mode (Fig. 3.8.) is pure torsion of the equipment section. It also represents graphically the effect of the isolator section in that it is completely uncoupled from the optical bench. The eighth mode (Fig. 3.9.), like mode #7, is completely uncoupled by the isolator section and represents pure second mode torsion of the optical bench.

3.2 CONCLUSIONS

We have presented the results of a simple model of an example large space structure. The results are in the form of mode shapes and natural frequencies as presented. In addition, because of the simplicity of the model one is able by inspection to ascertain the significant configurational drivers for each of the modes and frequencies and thereby be in the position to change them in an appropriate manner, thereby providing "management flexibility." It is through this process of developing and main-

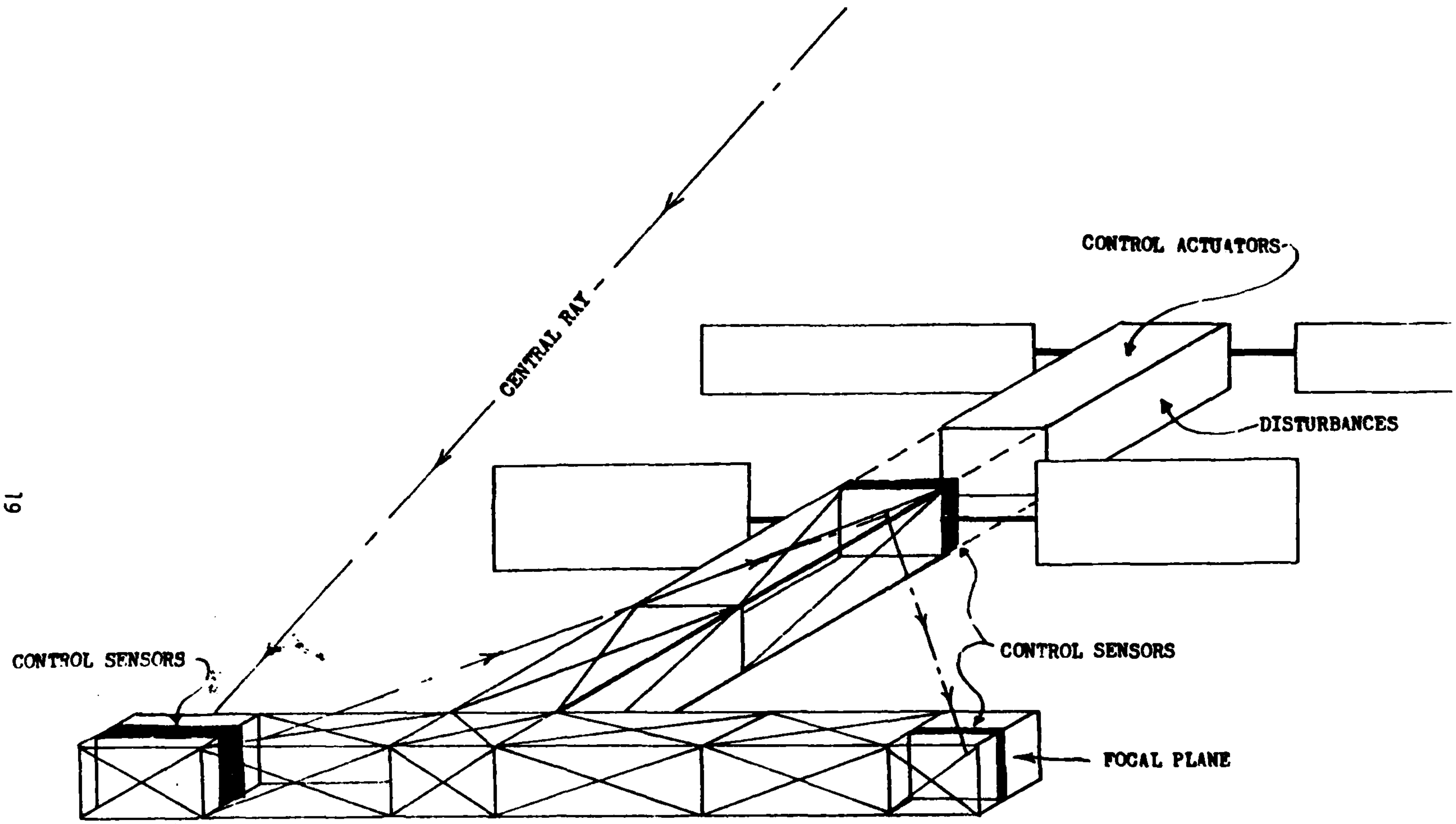
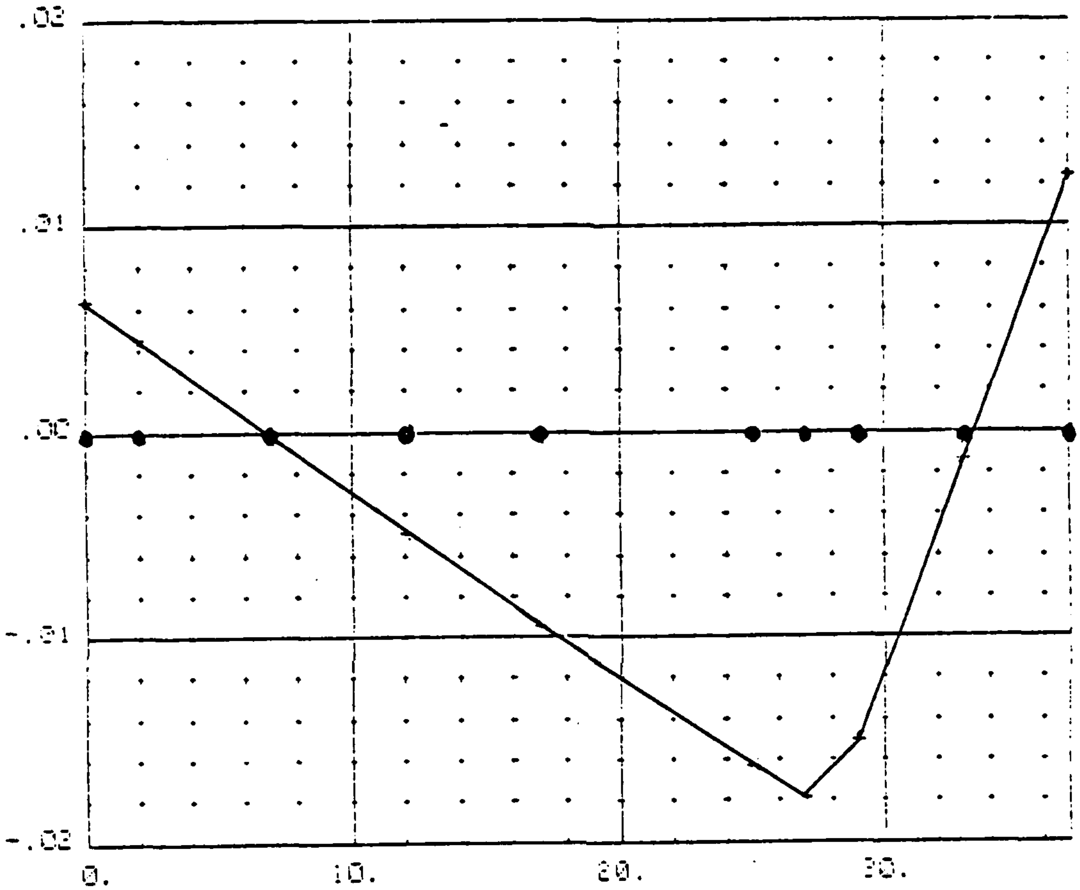


Fig. 3.1. SYSTEM CONFIGURATION



ωn=.1488 + - BENDING ● - TORSION

Fig. 3.2 First Bending Mode

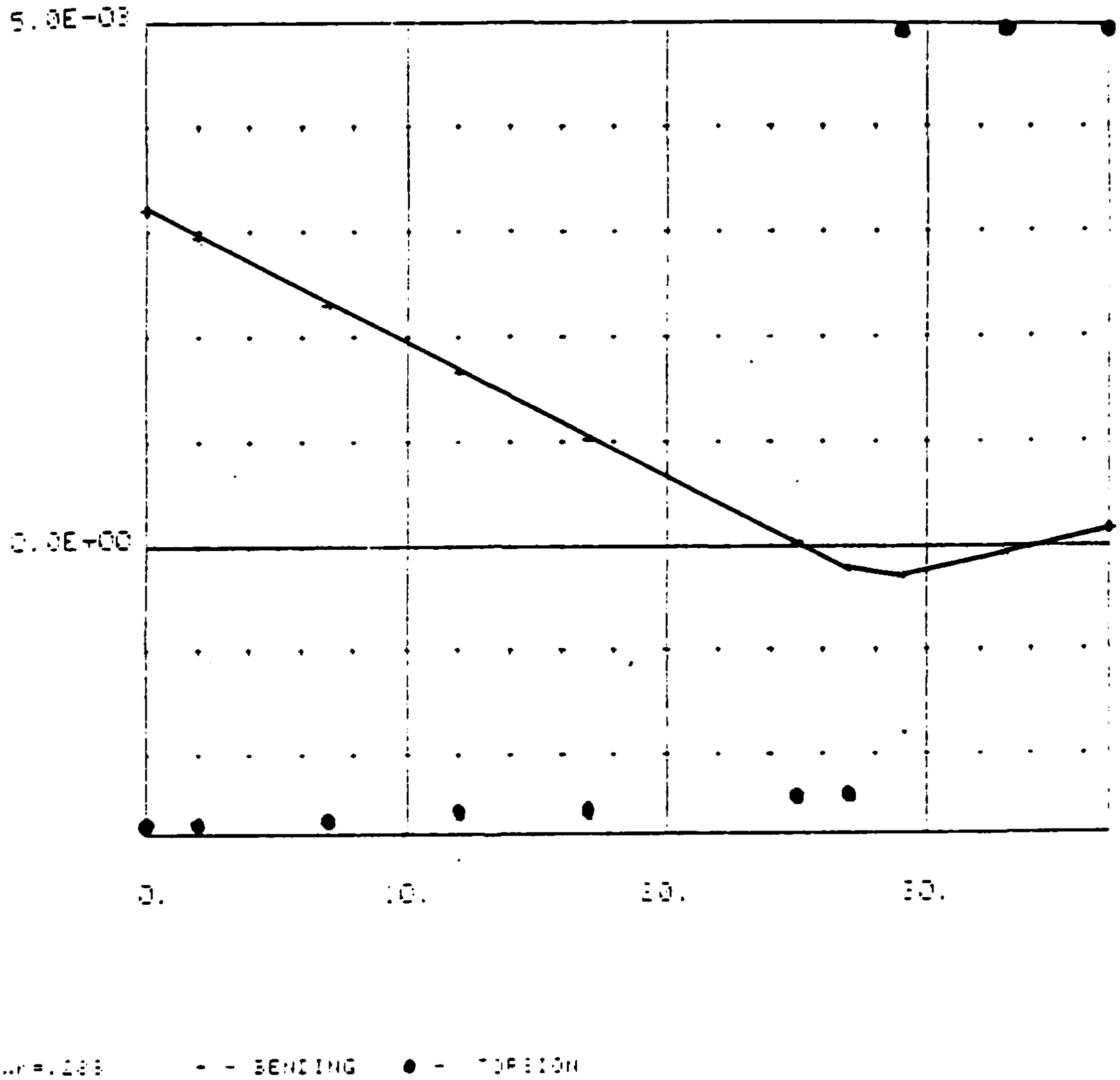


Fig. 3.3. Second Mode

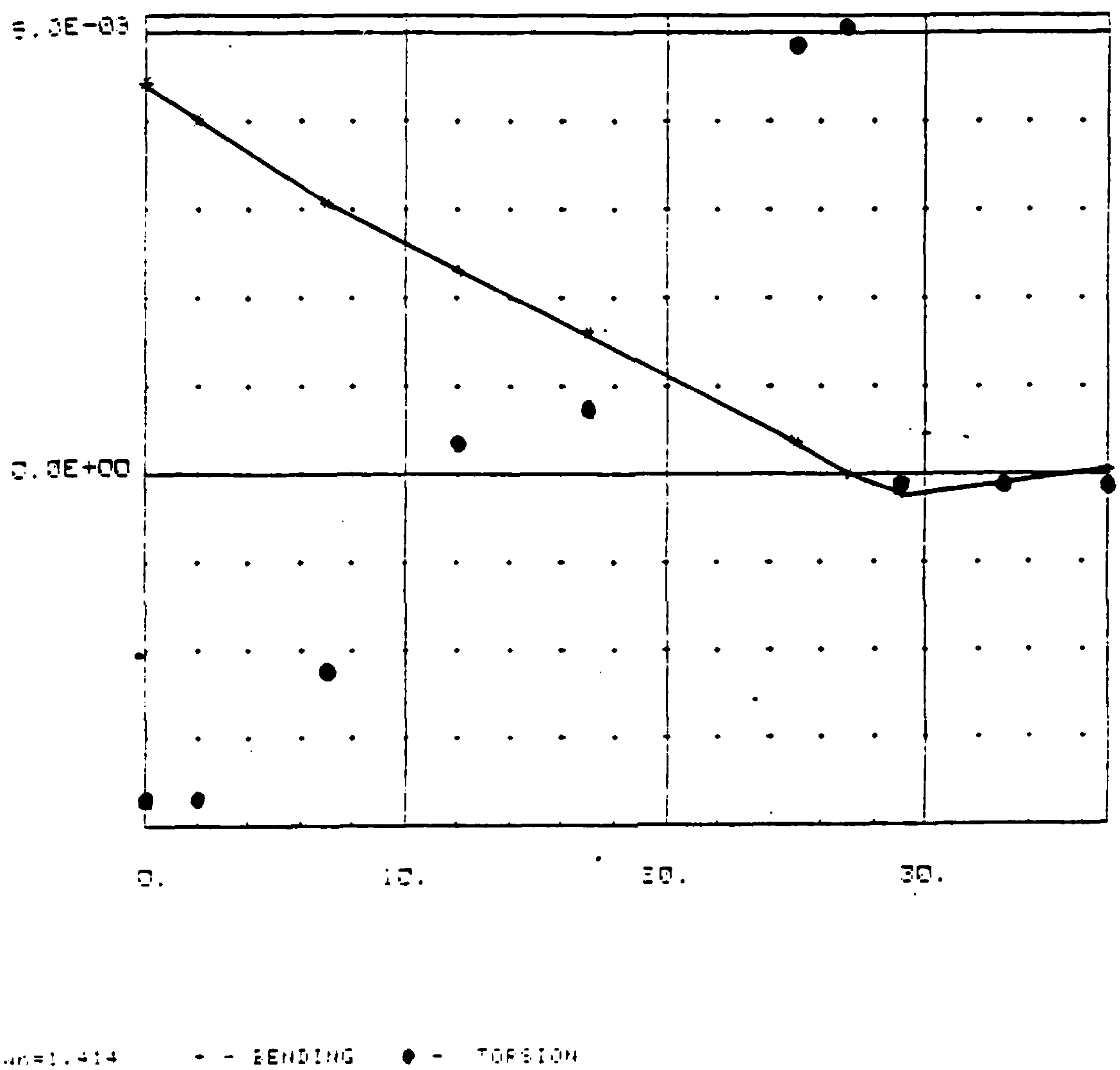


Fig. 3.4. Third Mode

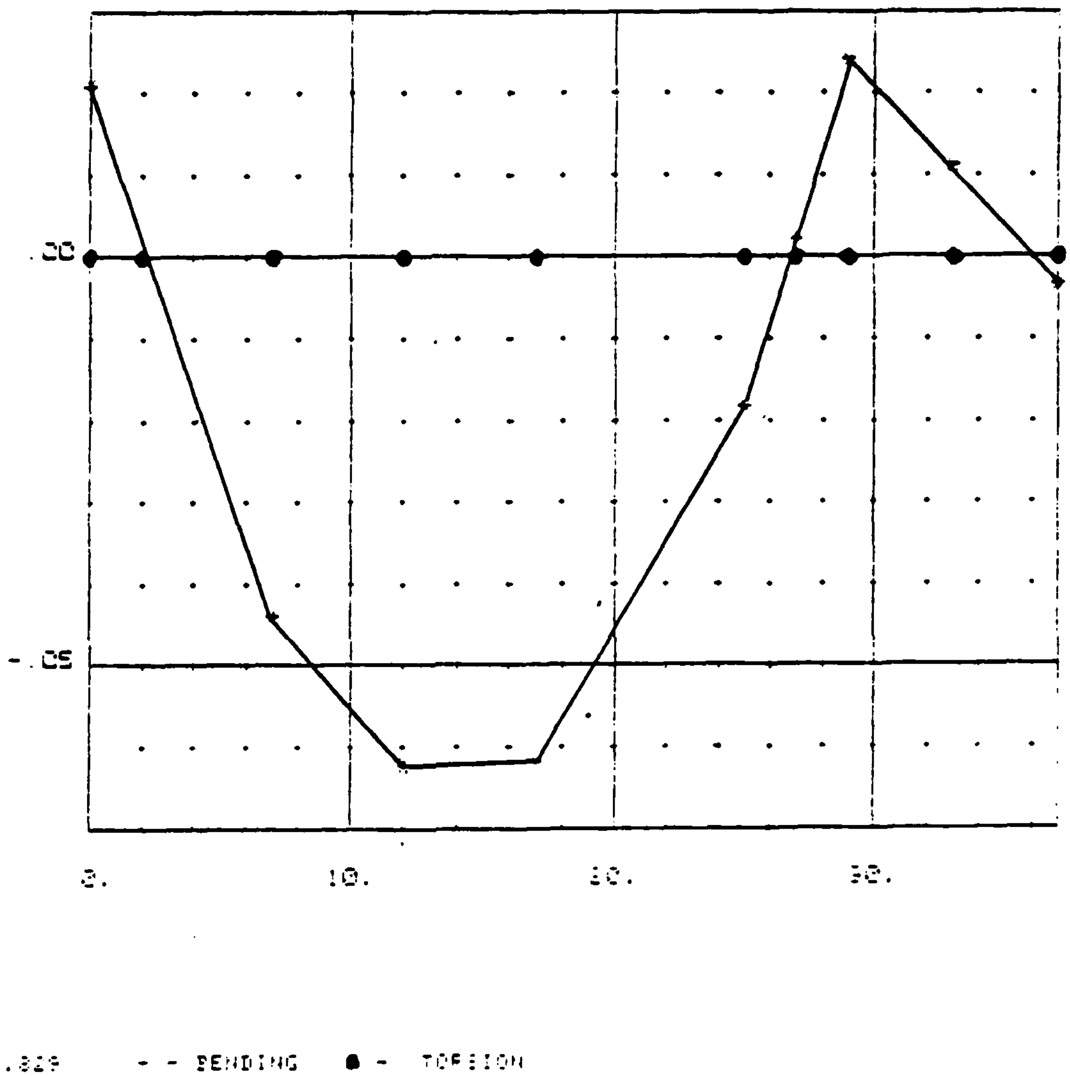


Fig. 3.5. Fourth Mode

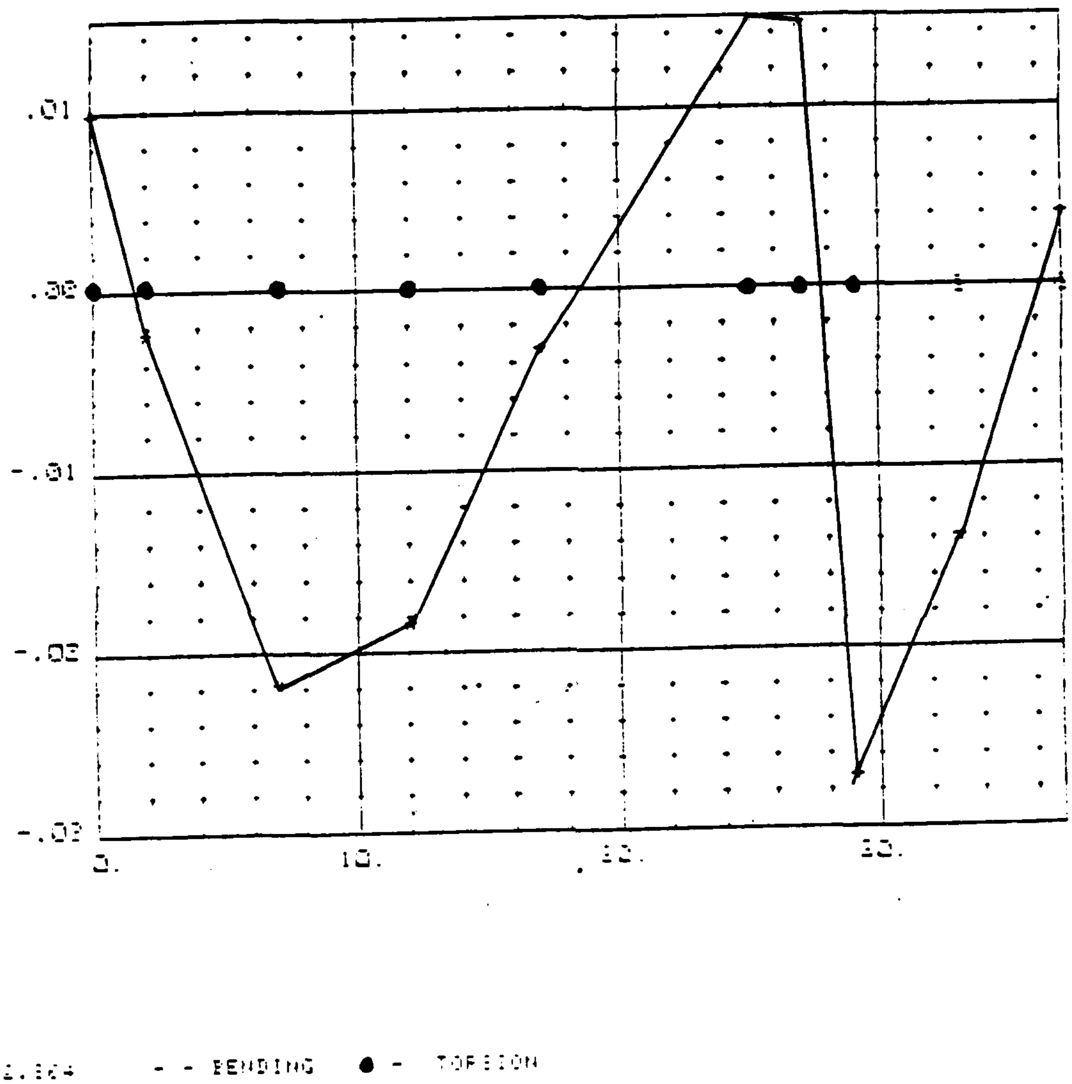


Fig. 3.6. Fifth Mode

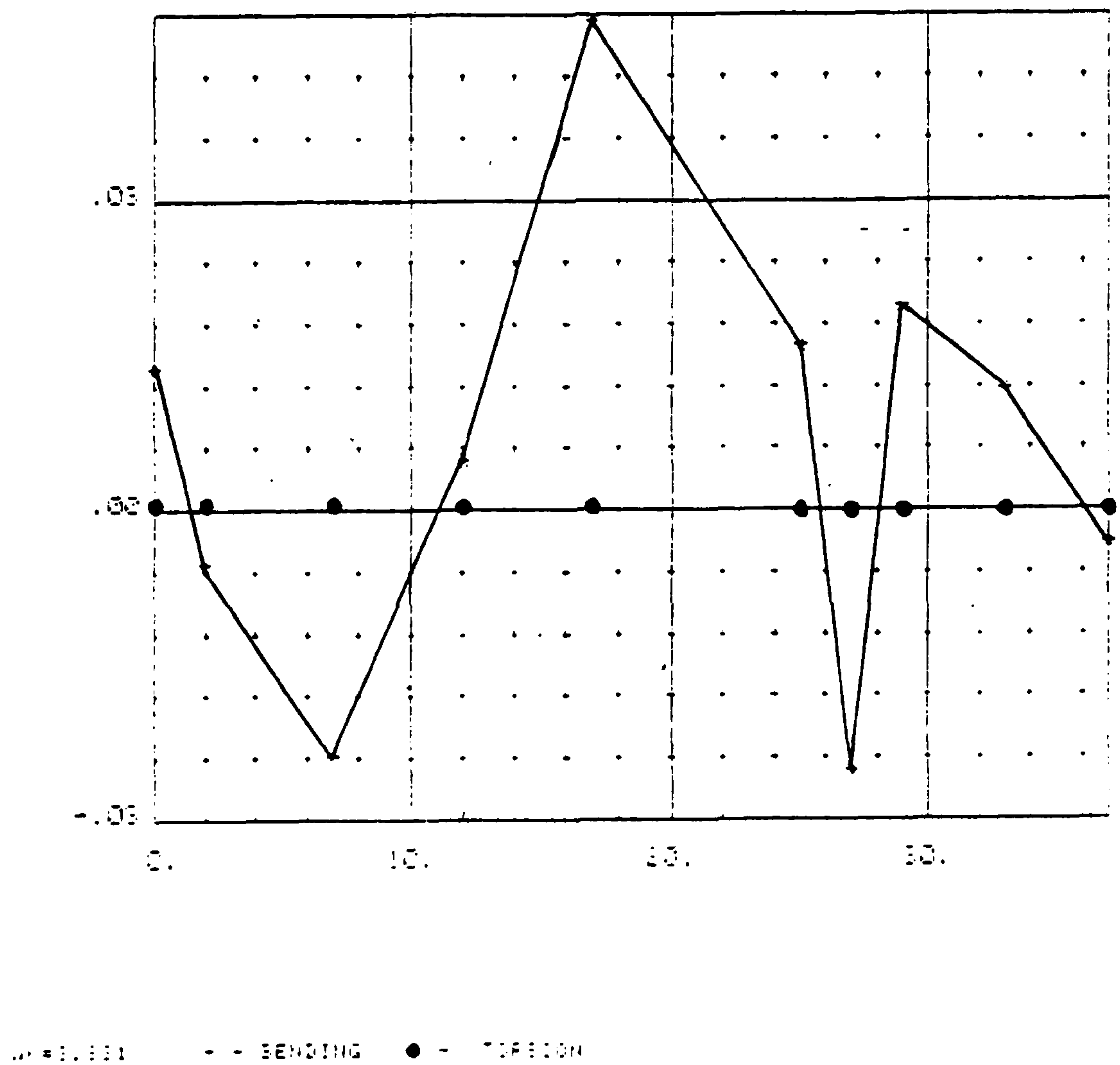


Fig. 3.7. Sixth Mode

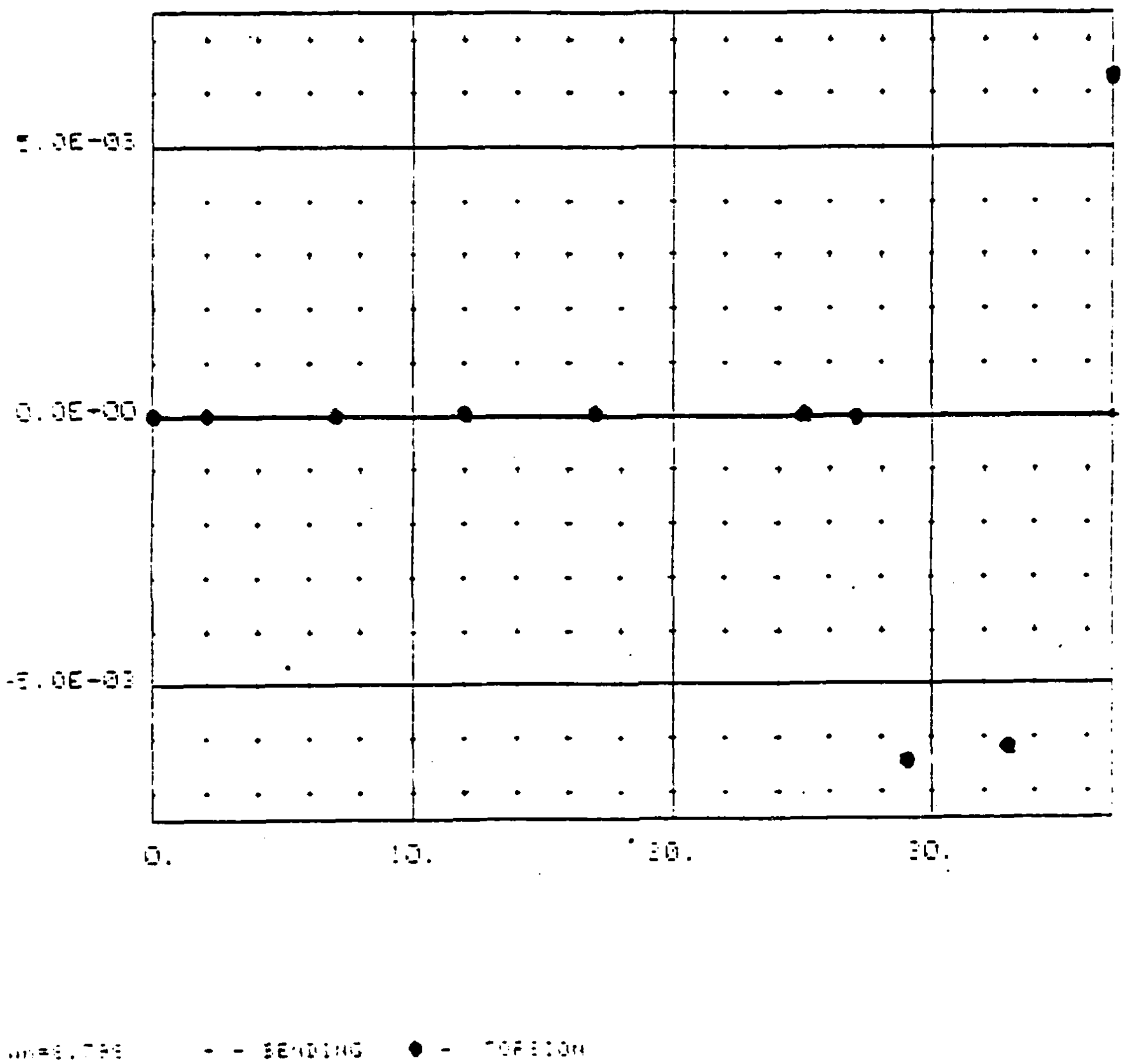


Fig. 3.8. Seventh Mode

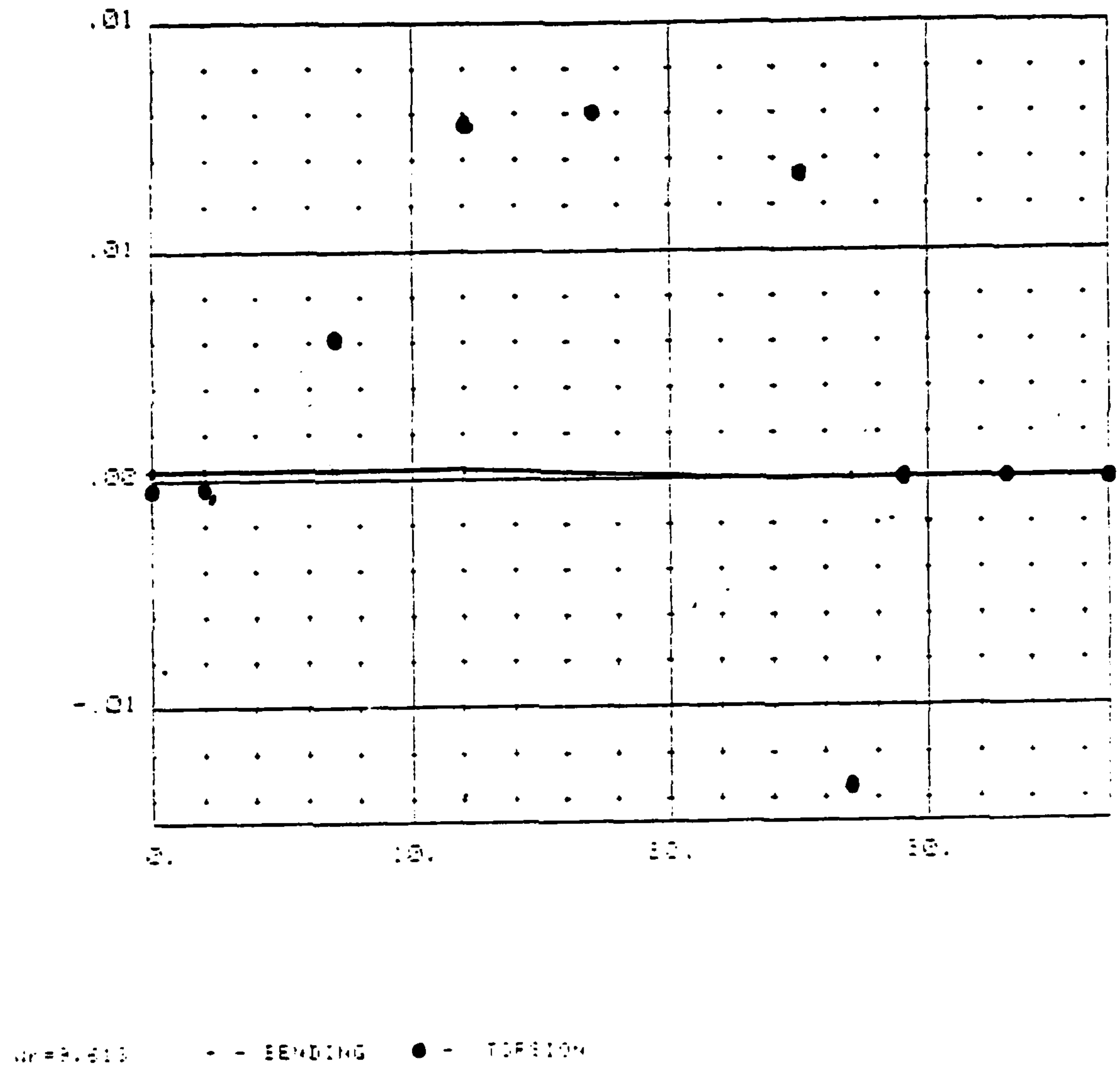


Fig. 3.9. Eighth Mode

taining an understanding of not only the control process but also the open loop plant characteristics that we believe is the most effective way to design control systems for vehicles. Therefore, it is also believed that if one is attempting to design a structure that would exercise control theorists, then one should pursue a similar course. We did not always have these giant computer programs with us and large space structures were flown very successfully in the past (for example, the SATURN IB and V launch vehicles and the SKYLAB manned satellite). Finally, of course, the program costs and time are minimal.

4.0 VALIDATION OF REDUCED-ORDER MODELS FOR CONTROL SYSTEM DESIGN

The concept of suboptimality is applied to testing validity of reduced-order models in design of feedback schemes for large-scale systems. Aggregation and singular value decomposition as model reduction techniques, are interpreted in the expansion-contraction framework, which is suitable for evaluation of suboptimality of closed-loop systems resulting from reduced order designs. The proposed validation procedure is applied to a control design of a large space structure.

4.1 INTRODUCTION

In system theory and applications, "model reduction" stands for a variety of more or less disparate concepts and techniques which have a common goal: to reduce the size of the mathematical model of a large system in order to simplify the design of control and estimation schemes. Due to the nature of the mathematical models involved in describing the motion of Large Space Structures (LSS), a large number of recent results in model reduction have been obtained in this field, with emphasis on the size and complexity of the controller to be implemented onboard the spacecraft (see, for example, [1] and [2]). Most of the proposed model reduction procedures has a common plan: First, the elements of a mathematical model are separated into two parts according to the significance of their contribution to the desired performance characteristic of the system (stability, optimality, frequency response, and the like). Second, the most significant part of the model is used to design the control scheme for the overall system. Finally, the scheme is implemented and the system performance is tested usually by an extensive simulation analysis. The emphasis on each phase of the model reduction plan varies from concept to concept, but the first phase involving the decomposition of the model has been the most considered area in the research on model reductions.

Almost exclusively, all the model reduction techniques are concerned with the open-loop behavior of the system and are, therefore, subject to the question of whether the reduced-order model would still be a good representation of the overall system after the feedback loop is closed. The answer to this question would most probably be positive if a reduced-order model were indeed a good approximation of the original system. Yet, it is difficult to say how good the approximation is without a comparison of the

resulting closed-loop system with the one that would be obtained under ideal conditions when a full controller is designed using the entire model of the system.

In order to resolve the validation problem of model-reduction schemes, we note that there are two distinct simplifications attempted by the methods: To obtain a simple control law which is suitable for implementation and to reduce the computations involved in the control design. Since the off-line computations needed to determine the best control law for dynamic systems of considerable size may well be within the capabilities of today's computers, the dimensionality problem is not as severe as the compatibility problem of the control structure with the design constraints. For this reason, an appropriate approach to the model reduction problem is to concentrate on construction of suitable control laws and, subsequently, compare the resulting closed-loop system with the optimal off-line model used as reference. This approach to model reduction is in the spirit of suboptimal design schemes developed in the context of large-scale systems [3, 4], and the major objective of this paper is to adapt these suboptimality schemes for use in model-reduction problems. This reformulation, in turn, opens up a real possibility for future use of the model-reduction approach in the design of modern decentralized control schemes for large space structures composed of interconnected systems.

The organization of this section is as follows:

In the Section 4.2 , we briefly summarize the concept of suboptimality which provides a basis for the subsequent development. In Section 4.3, we outline the inclusion principle, which constitutes an appropriate mathematical framework for perfect model reduction. In this section, we also study the role of inclusion principle in the context of suboptimal control, and show that when perfect model reduction is possible, then the suboptimal control

obtained from the reduced-order model is actually optimal for the original overall system. In Section 4.4, we elaborate on various approximate model reduction schemes. From the point of view of suboptimality, these schemes appear as deviations from perfect model reductions obtained via the inclusion principle. In Section 4.5, a model reduction technique based upon the balanced representations, is applied to a model of a large space structure. Finally, in conclusion, we discuss possible extensions of the suboptimal approach to more elaborate problems of LSS.

4.2 SUBOPTIMAL CONTROL

Let us consider a linear system S described as

$$S:\ \dot{x} = Ax + Bu\ ,\quad x(0) = x_0 \tag{2.1}$$

where $x(t) \in R^n$ is the state and $u(t) \in R^m$ is the input of S, and the matrices A and B are constant and of appropriate dimensions. With S we associate a quadratic performance index

$$J = \int_0^\infty (x^TQx + u^TRu)dt\ , \tag{2.2}$$

where Q is a symmetric nonnegative definite matrix and R is a symmetric positive definite matrix, both matrices being constant and of appropriate dimensions. The system S and the cost functional J form a standard optimization problem $\{S, J\}$, the solution of which is given as

$$u^o = -K^o x\ , \tag{2.3}$$

where the gain matrix K^o is

$$K^o = R^{-1}B^TP\ , \tag{2.4}$$

and P is the solution of the Riccati equation

$$A^TP + PA - PBR^{-1}B^TP + Q = 0\ . \tag{2.5}$$

Application of the control u^o to S results in the optimal closed-loop system

$$S^o:\ \dot{x} = (A-BK^o)x\ , \tag{2.6}$$

and yields the optimal cost $J^o(x_0) = J(x_0, K^o)$,

$$J^o(x_0) = x_0^TPx_0\ . \tag{2.7}$$

Now, suppose instead of the optimal control u°, we apply another control

$$u = -Kx , \tag{2.8}$$

where the gain matrix K is obtained by some rational procedure which is different from the above optimal solution of $\{S, J\}$. Then, the control u of (2.8) produces a cost $J^{\bullet}(x_0) = J(x_0, K)$, which is, in general, larger than the optimal cost $J^{\circ}(x_0)$. In fact, the cost $J^{\bullet}(x_0)$ may very well be infinite if the control in (2.8) is not stabilizing. We now state the following [4]:

(2.9) DEFINITION. *The control law* (2.8) *is said to be suboptimal with degree* μ *for* S *with respect to* S° *if there exists a positive number* μ *such that*

$$J^{\bullet}(x_0) \le \mu^{-1} J^{\circ}(x_0) , \tag{2.10}$$

for all x_0.

To derive conditions for suboptimality and compute the suboptimality index μ, we note that if the control (2.8) is stabilizing for S, then the suboptimal cost is given by

$$J^{\bullet}(x_0) = x_0^T H x_0 , \tag{2.11}$$

where H is the unique, symmetric, and positive definite solution of the Liapunov equation

$$(A-BK)^T H + H(A-BK) + Q + K^T RK = 0 . \tag{2.12}$$

This leads to [5]:

(2.13) THEOREM. *If the control law* (2.8) *is stabilizing for* S, *then it is suboptimal for* S *with degree*

$$\mu = \lambda_M^{-1}(HP^{-1}) , \tag{2.14}$$

where λ_M *denotes the largest eigenvalue of the indicated matrix.*

An alternative suboptimality criterion, which does not require a test for stability, is provided by the following [4]:

(2.15) THEOREM. *The control law* (2.8) *is suboptimal for* S *with degree* μ *if the matrix*

$$F(\mu) = (K^o-K)^T R(K^o-K) - (1-\mu)(Q + K^T RK) \tag{2.16}$$

is nonnegative definite.

We note that to compute the suboptimality index μ using Theorem (2.13) or (2.15), we have to compute the solution of the problem $\{S, J\}$. Although we know the optimal control (2.3), we may prefer to apply the suboptimal control (2.8). One important reason might be that the optimal control is difficult to implement in the actual design. For example, we may choose the output feedback

$$u = -Fy , \tag{2.17}$$

where

$$y = Cx \tag{2.18}$$

is the only available measurement with the matrix C having a full row rank. In this case, a reasonable choice for K would be

$$K = K^o C^T(CC^T)^{-1}C , \tag{2.19}$$

which is the projection of K^o on the row space of the output matrix C. We do not gain anything computationally, but only satisfy the design constraints on the control structure. Another way to generate the gain matrix K in (2.8), which satisfies the information structure constraint, is to re-

place by zeros those elements of K^o that correspond to states which are not available for feedback. Following [4], both types of control are referred to as *degenerate control.*

Alternatively, the control (2.8) can be generated from the solution of a smaller problem $\{\bar{S}, \bar{J}\}$ where the system

$$\bar{S}: \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u , \quad \bar{x}(0) = \bar{x}_0 \tag{2.20}$$

represents a reduced order model of S , that is, $\bar{x}(t) \in R^{\bar{n}}$ and $\bar{n} \leq n$. The cost

$$\bar{J} = \int_0^\infty (\bar{x}^T\bar{Q}\bar{x} + u\bar{R}u)dt , \tag{2.21}$$

is obtained from the cost J in a manner compatible with the reduction of S to $\bar{S}$. Usually, $\bar{S}$ can be chosen as a projection of S onto some subspace of the state space [5 , 6] defined by a singular transformation

$$\bar{x} = Ux . \tag{2.22}$$

Now, if the optimal solution of the problem $\{\bar{S}, \bar{J}\}$ is obtained as

$$u = -\bar{K}^o\bar{x} , \tag{2.23}$$

then the control

$$u = -\bar{K}^oUx \tag{2.24}$$

can be a reasonable choice for the original system S . Suboptimality of the control (2.24) can be tested using Theorem (2.13) or (2.15) with $K = \bar{K}^oU$.

Another advantage of obtaining the suboptimal control by the solution of a smaller optimization problem $\{\bar{S}, \bar{J}\}$, is a possibility to compare the suboptimal cost $J^\circledast(x_0) = J(x_0, \bar{K}^oU)$ with the optimal cost $\bar{J}^o(\bar{x}_0) = \bar{J}(\bar{x}_0, \bar{K}^o)$ of pair $\{\bar{S}, \bar{J}\}$. Of course, $\bar{x}_0$ should be chosen as the projection of x_0 , that is, $\bar{x}_0 = Ux_0$. If this comparison can be worked out, then we can

avoid solving the original problem $\{S, J\}$, thus providing for both computational and implementational simplifications. In the context of suboptimality [4, 5], taking the optimal small system $\bar{S}^o$ as reference in the definition of suboptimality, is the dual of taking the original system S^o as the reference system, in the sense that in the latter case the control is considered to be perturbed from the optimal value, while in the former case the system is perturbed by the residuals which were eliminated by the projection.

4.3 THE INCLUSION PRINCIPLE

In using a reduced order model to build a controller, it is important to know if the behavior of the original system can be predicted by the model. The *inclusion principle* of [5 , 6] provides conditions under which the reduced order model represents perfectly the motion of the original system. In this section, we briefly summarize the principle and discuss its implications.

Let us reconsider the two systems S and $\bar{S}$, which are described now as

$$S: \dot{x} = Ax + Bu , \quad x(0) = x_0$$
$$y = Cx \tag{3.1}$$

and

$$\bar{S}: \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u , \quad \bar{x}(0) = \bar{x}_0$$
$$\bar{y} = \bar{C}\bar{x} \tag{3.2}$$

where the outputs $y(t) \in R^{\ell}$ and $\bar{y}(t) \in R^{\ell}$ are added to the previous descriptions (2.1) and (2.20). The solutions of (3.1) and (3.2) are denoted by $x(t, x_0; u)$ and $\bar{x}(t, \bar{x}_0; u)$, respectively. We have:

(3.3) DEFINITION. *The system* S *of* (3.1) *is said to include the system* $\bar{S}$ *of* (3.2) *if there exist matrices* U *and* V *of order* $\bar{n} \times n$ *and* $n \times \bar{n}$ *such that* $UV = I_{\bar{n}}$ *and*

$$\bar{x}(t, \bar{x}_0; u) = Ux(t, V\bar{x}_0; u) \tag{3.4}$$

and

$$\bar{y}(t) = y(t) \tag{3.5}$$

for all $t, \bar{x}_0$, *and* $u(t)$.

Definition (3.3) implies that if S includes $\bar{S}$, then the system $\bar{S}$ contains all the information about the system S . If S includes $\bar{S}$, then S is said to be an *expansion* of $\bar{S}$, and $\bar{S}$ is a *contraction* of S . In

(3.3), $I_{\bar{n}}$ denotes the $\bar{n} \times \bar{n}$ identity matrix. Obviously, an interesting question is: Under what conditions S includes $\bar{S}$? This is provided by the following:

(3.6) THEOREM. *S includes $\bar{S}$ if and only if there exist matrices* U *and* V *such that* $UV = I_{\bar{n}}$ *and*

$$UA^iV = \bar{A}^i, \; UA^{i-1}B = \bar{A}^{i-1}\bar{B}, \; CA^{i-1}V = \bar{C}\bar{A}^{i-1}, \; CA^{i-1}B = \bar{C}\bar{A}^{i-1}\bar{B} \qquad i = 1, 2, \ldots, \bar{n} . \tag{3.7}$$

Let us consider some special cases. First, suppose there exists an $\bar{n} \times n$ matrix U with full row rank such that

$$UA = \bar{A}U \; , \quad UB = \bar{B} \; , \quad C = \bar{C}U \; . \tag{3.8}$$

Then, it is easy to show that the conditions in (3.7) are satisfied for any right inverse V of U , that is, for any V such that $UV = I_{\bar{n}}$. Therefore, if the conditions (3.8) are satisfied for some U , then $\bar{S}$ is a contraction of S and, hence, (3.4) and (3.5) hold. But, in this special case, there is a stronger relation between the solutions of S and $\bar{S}$, namely,

$$\bar{x}(t, Ux_0; u) = Ux(t, x_0; u) \; , \tag{3.9}$$

for all t, x_0 , and $u(t)$. In other words, whatever the initial state x_0 , a projection of the solution of S can be recovered from a solution of $\bar{S}$. In this case, $\bar{S}$ is an *aggregation* of S in the sense of Aoki [7].

As another special case, suppose there exists an $n \times \bar{n}$ matrix V with full column rank such that

$$AV = V\bar{A} \; , \quad B = V\bar{B} \; , \quad CV = \bar{C} \; . \tag{3.10}$$

Then, for any left inverse U of V satisfying $UV = I_{\bar{n}}$, conditions (3.7) hold, and $\bar{S}$ is a contraction of S . In this case, too, a stronger relation holds,

$$V\bar{x}(t,\ \bar{x}_0;\ u) = x(t,\ V\bar{x}_0;\ u) \quad . \tag{3.11}$$

which implies that solutions of S starting in a subspace of the state space of S can be recovered from the solutions of $\bar{S}$. In this case we say that $\bar{S}$ is a *restriction* of S.

Let us now broaden the inclusion principle to consider the optimization problems via model reduction. In the optimal control problem $\{S,\ J\}$, we let

$$Q = C^TC \tag{3.12}$$

for some matrix C and rewrite J as

$$J = \int_0^\infty (y^Ty + u^TRu)dt \tag{3.13}$$

where $y(t)$ may be taken as the output of S described by (3.1). Let us assume that the system S of (3.1) is aggregable to a system $\bar{S}$ described by (3.2). With $\bar{S}$ we associate the cost

$$\bar{J} = \int_0^\infty (\bar{y}^T\bar{y} + u^TRu)dt \ . \tag{3.14}$$

We choose a control

$$u = -\bar{K}\bar{x} \ , \tag{3.15}$$

which is not necessarily optimal for $\{\bar{S},\ \bar{J}\}$, and apply to $\bar{S}$. Under the assumption that it is stabilizing, the control (3.15) yields the cost given by

$$\bar{J}(\bar{x}_0,\ \bar{K}) = \bar{x}_0^T\bar{H}\bar{x}_0 \tag{3.16}$$

where $\bar{H}$ is the unique, symmetric, and positive definite solution of the Liapunov matrix equation

$$(\bar{A}-\bar{B}\bar{K})^T\bar{H} + \bar{H}(\bar{A}-\bar{B}\bar{K}) + \bar{C}^T\bar{C} + \bar{K}^TR\bar{K} = 0 \ , \tag{3.17}$$

Multiplying (3.17) by U^T from the left and by U from the right, and using the aggregation conditions (3.8), we get

$$(A-BK)^T H + H(A-BK) + C^T C + K^T RK = 0 \,, \tag{3.18}$$

where

$$K = \bar{K}U \,, \quad H = U^T \bar{H} U \,. \tag{3.19}$$

Comparing (3.18) with (2.12), and keeping in mind that $Q = C^T C$, we conclude that

$$J(x_0, \bar{K}U) = x_0^T U^T \bar{H} U x_0 = \bar{J}(Ux_0, \bar{K}) \,. \tag{3.20}$$

In particular, if $\bar{K} = \bar{K}^o$, the optimal feedback matrix for the problem $\{\bar{S}, \bar{J}\}$, then (3.20) gives

$$J(x_0, \bar{K}^o U) = \bar{J}(Ux_0, \bar{K}^o) = \bar{J}^o(Ux_0) \,. \tag{3.21}$$

Since $\bar{J}^o(Ux_0)$ is the global minimum of $\bar{J}$, (3.21) further implies that

$$\bar{K}^o U = K^o \,, \tag{3.22}$$

and

$$J(x_0, \bar{K}^o U) = J^o(x_0) = \bar{J}^o(Ux_0) \,, \tag{3.23}$$

which establish the relation between the optimal controls and costs of the two optimization problems $\{S, J\}$ and $\{\bar{S}, \bar{J}\}$ when $\bar{S}$ is an aggregation of S. This result could have been achieved directly by comparing the Riccati equations corresponding to the two problems. We preferred the above derivation, because we can reproduce it in the suboptimality context outlined in the next section.

Similarly, if $\bar{S}$ is a restriction of S, we can show that

$$J(V\bar{x}_0, K) = \bar{J}(\bar{x}_0, KV) \,, \tag{3.24}$$

provided that costs are finite. In particular,

$$K^{o}V = \bar{K}^{o} \tag{3.25}$$

and

$$J^{o}(V\bar{x}_{0}) = \bar{J}^{o}(x_{0}) \ , \tag{3.26}$$

which relates the optimal control and costs for the two optimization problems.

4.4 SUBOPTIMALITY OF REDUCED ORDER MODELS

One of the difficulties in applying the inclusion principle to model reduction is the fact that it may be overly restrictive and conditions for a given system S to have a contraction $\tilde{S}$ do not hold. A natural way to resolve this problem is to introduce an approximate contraction procedure and evaluate the suboptimality of the end result.

It is clear from the aggregation conditions (3.8) that the system S is aggregable to a smaller system $\bar{S}$ if and only if it is unobservable. Thus, if the problem $\{S, J\}$ is such that all the state variables are observed in the performance index J, then it cannot be reduced to a smaller problem. In order to produce an approximate aggregation, we split the system matrix A into two matrices as

$$A = A_N + A_S \,. \tag{4.1}$$

By choosing the "surplus matrix" A_S appropriately, the "new" matrix $A_N = A - A_S$ can be made such that the system

$$\begin{aligned} S_N\colon\ \dot{x} &= A_N x + Bu \\ y &= Cx \end{aligned} \tag{4.2}$$

is aggregable to a smaller system

$$\begin{aligned} \bar{S}\colon\ \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ \bar{y} &= \bar{C}\bar{x} \quad . \end{aligned} \tag{4.3}$$

The aggregation conditions (3.8) require that

$$U(A - A_S) = \bar{A}U \ , \ C = \bar{C}U \tag{4.4}$$

for some U, $\bar{A}$, and $\bar{C}$. The choice of U is arbitrary provided the second equation in (4.4) is satisfied by some $\bar{C}$. An obvious choice is $U = C$, $\bar{C} = I$, which produces $\bar{S}$ of smallest dimension. This may, however, require

the elements of A_S to be large compared to those of A, which is not desirable since we want to make $\bar{S}$ "close" to S. Alternatively, making use of the concept of chain aggregation [8] , we can choose $U = [C^T\ U_C^T]^T$, $\bar{C} = [I\ \ 0]$, where U_C is such that U has full row rank. In this case, a suitable choice of U_C can make it possible to satisfy (4.4) with A_S sufficiently small.

With U fixed, from (4.4) we get

$$UA_S = UA - \bar{A}U , \tag{4.5}$$

which suggests that $\bar{A}$ should be chosen as

$$\bar{A} = UA\,U^T(UU^T)^{-1} \tag{4.6}$$

in order to minimize $||UA_S||$. With $\bar{A}$ as in (4.6), (4.5) reduces to

$$UA_S = UA\ [I - U^T(UU^T)^{-1}U] , \tag{4.7}$$

the minimum-norm solution of which is

$$A_S = U^T(UU^T)^{-1}UA\,[I - U^T(UU^T)^{-1}U] . \tag{4.8}$$

In summary, we showed that by subtracting a matrix A_S from the nominal matrix A , we can produce an approximate aggregate $\bar{S}$ of S for some U and $\bar{C}$. The matrix $\bar{A}$ of $\bar{S}$ can be determined from (4.6) and $\bar{B} = UB$. We can then proceed to solve the reduced order problem $\{\bar{S}, \bar{J}\}$ and produce the low-order gain matrix $\bar{K}^o$. Then, the optimization problem $\{S , J\}$ can be solved to serve as a reference for suboptimality of the system S driven by the control $u = \bar{K}^o Ux$ of (2.24), which is obtained from the low-order design $\bar{K}^o$. Suboptimality conditions are provided by Theorem (2.13) or (2.15).

An alternative to the approximate aggregation considered above is to aggregate the weighting matrix C^TC of the state of S in the performance index J, as proposed by Aoki [7]. This is done by choosing an aggregation matrix U with full row rank, such that

$$UA = \bar{A}U \tag{4.9}$$

for some matrix $\bar{A}$, and computing $\bar{B}$ as

$$\bar{B} = UB \ . \tag{4.10}$$

Conditions (4.9) and (4.10) are part of the aggregation conditions in (3.8). However, since we assumed that S is not perfectly aggregable to a smaller system, there is no $\bar{C}$ that satisfies the last aggregation condition, namely, $C = \bar{C}U$. Here, we deviate from perfect aggregation, and choose $\bar{C}$ such that $||C-\bar{C}U||$ is minimum, i.e.,

$$\bar{C} = CU^T(UU^T)^{-1} \ . \tag{4.11}$$

The matrices $\{\bar{A}, \bar{B}, \bar{C}\}$ of (4.9)-(4.11) constitute an approximate aggregate system $\bar{S}$ of (4.4), with which we associate the cost $\bar{J}$ in (3.15).

Now, consider the Riccati equation

$$\bar{A}^T\bar{P} + \bar{P}\bar{A} - \bar{P}\bar{B}R^{-1}\bar{B}^T\bar{P} + \bar{C}^T\bar{C} = 0 \tag{4.12}$$

associated with the aggregate problem $\{\bar{S}, \bar{J}\}$. Multiplying (4.12) by U^T from the left and by U from the right, and using (4.9)-(4.11) we obtain

$$A^T(U^T\bar{P}U) + (U^T\bar{P}U)A - (U^T\bar{P}U)BR^{-1}B^T(U^T\bar{P}U) + \Pi C^TC\Pi = 0 \tag{4.13}$$

where the matrix

$$\Pi = U^T(UU^T)^{-1}U \ , \tag{4.14}$$

is a projection on R^n. Comparing (4.13) with the Riccati equation (2.5)

of the original problem $\{S, J\}$, we observe that $U^T\bar{P}U$ can be considered as an approximate of P provided $C\Pi$ is close to C. Thus,

$$K = \bar{K}^o U = R^{-1}\bar{B}^T\bar{P}U = R^{-1}B^T(U^T\bar{P}U) \tag{4.15}$$

is an approximate to the optimal feedback K^o in (2.4), the suboptimality of which can be tested using Theorem (3.13) or (3.15).

We note that condition (4.9) requires that U consist of reciprocal eigenvectors of A, and $\bar{A}$ retain those eigenvalues of A that correspond to these eigenvectors. Thus, the aggregation of the performance index as explained above consists of choosing an eigenspace of A and modifying the performance index to reflect the contribution of the selected modes to the cost. In this sense, this type of approximate aggregation is closely related to Skelton's model reduction scheme through modal cost analysis [9, 10]. For a given order $\bar{n}$ of the reduced order model, the $\bar{n}$ modes that contribute most to the cost J can be determined by computing U such that $||C-C\Pi||$ is minimum subject to (4.9), where Π is given by (4.14). However, whether the modes determined by this U matrix will be the most significant ones in the closed-loop system is not clear. One way of deciding on the best approximate aggregation might be to compute the degree of suboptimality μ for each of the $n!/(n-\bar{n})!\bar{n}!$ possible choices of U, and to choose the one with minimum μ. However, noting that the contribution of a mode to the closed-loop cost depends on how controllable as well as how observable it is, a more refined model reduction procedure can be developed in the suboptimality framework via singular-value decomposition.

The singular-value decomposition of a system using balanced representation has been developed by Moore [11] and later studied in detail by others (e.g., see [12, 13]). In simple terms, a balanced system is one which is as control-

lable as it is observable, where the measure of controllability and observability is provided by the singular values [11] of the respective grammians.

To illustrate the idea, suppose S is stable and let

$$\Sigma_C = \int_0^\infty e^{At} BB^T e^{A^T t} dt \ , \tag{4.16}$$

and

$$\Sigma_O = \int_0^\infty e^{A^T t} C^T C e^{At} dt \ . \tag{4.17}$$

denote the controllability and observability grammians of S respectively. Now consider the product $\Sigma_O \Sigma_C$. Assuming S is controllable and observable (otherwise, a preliminary procedure can be used to reduce the system order by deleting uncontrollable and/or unobservable part), Σ_O and Σ_C are positive definite matrices, and therefore, all the eigenvalues of $\Sigma_O \Sigma_C$ are positive. Let these eigenvalues be denoted by σ_i^2, with $\sigma_1 \geq \sigma_2 \geq \dots \sigma_n > 0$. The positive square roots σ_i of the eigenvalues of $\Sigma_O \Sigma_C$ are called the singular-values of the system S .

It was shown by Moore [11] that, there always exists a coordinate frame in which

$$\Sigma_C = \Sigma_O = \Sigma = \text{diag}\ (\sigma_1, \sigma_2, \dots, \sigma_n) \ . \tag{4.18}$$

The representation of S in this coordinate frame is said to be balanced.

Now, suppose the system S has a balanced representation, and let

$$\Sigma = \text{diag}\ (\Sigma_1, \Sigma_2) \ , \tag{4.19}$$

where

$$\begin{aligned} \Sigma_1 &= \text{diag}\ (\sigma_1, \sigma_2, \dots, \sigma_{\bar{n}}) \ , \\ \Sigma_2 &= \text{diag}\ (\sigma_{\bar{n}+1}, \sigma_{\bar{n}+2}, \dots, \sigma_n) \ . \end{aligned} \tag{4.20}$$

Let us consider the corresponding partition of $\{A, B, C\}$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} . \tag{4.21}$$

If $\sigma_{\bar{n}} >> \sigma_{\bar{n}+1}$ in (4.20), then we can consider $\{A_{11}, B_1, C_1\}$ as the most controllable and observable part of the system S. Thus, the $\bar{n}$-th order system $\bar{S} = \{\bar{A}, \bar{B}, \bar{C}\}$ with $\bar{A} = A_{11}$, $\bar{B} = B_1$, $\bar{C} = C_1$, can be taken as a good reduced order model for S. In fact, it is easy to see from (4.19) that $\bar{S}$ is a good approximation to S in the sense that the differences between the controllability and observability grammians of $\bar{S}$ and S are minimum in the norm square sense.

Once a reduced order model $\bar{S}$ of the system S is obtained through the singular-value decomposition of a balanced representation of S, a suboptimal control can be designed for S following the procedure of Section 2. This approach to model reduction is particularly promising since it allows for generating systematically a sequence of reduced order models consisting of smaller and smaller portions of the A, B, C matrices in (4.21). Computing the degree of suboptimality at each step one can compromise between the degree of suboptimality and the order of the reduced model.

Finally we note that order reduction through singular-value decomposition can be considered as a combination of the previous two approaches, where we first subtract from the A matrix in (4.21) the surplus matrix

$$A_S = \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix}, \tag{4.22}$$

so that with $U = [I_{\bar{n}} \quad 0]$, $U(A - A_S) = \bar{A}\, U$, and then choose $\bar{C}$ as in (4.11), which yields $\bar{C} = C_1$. However, this procedure is not justified unless S is balanced.

4.5 APPLICATION

To illustrate the validation of model reduction schemes using the concept of suboptimality, we consider the feedback design of a large space structure described in [14]. The mathematical model is represented as a set of nine second order equations

$$\ddot{z}_i + d_i\dot{z}_i + \omega_i^2 z_i = b_i^T u \ , \quad i = 1, 2, \ldots , 9 \ , \tag{5.1}$$

where $z_i(t)$ are the vibration modes with damping d_i and frequency ω_i, and $u(t)$ is the two dimensional control input such that $b_i^T u$ is the scalar control torque on the i-th mode. The parameters d_i, ω_i and b_i^T are tabulated in Table 1.

Letting $x_i = [z_i \ \dot{z}_i]^T$, and $x = [x_1^T, x_2^T, \ldots , x_9^T]^T$, (5.1) can be written compactly as

$$\dot{x} = Ax + Bu \ , \tag{5.2}$$

where $A = \text{diag}\ \{A_1, A_2, \ldots , A_9\}$ with

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -d_i \end{bmatrix} , \tag{5.3}$$

and $B = \left[B_1^T, B_2^T, \ldots , B_9^T\right]^T$ with

$$B_i = \begin{bmatrix} 0 \\ b_i^T \end{bmatrix} . \tag{5.4}$$

We choose the output matrix to be of the form $C = \text{diag}\ \{c_1^T, c_2^T, \ldots , c_9^T\}$, with

$$c_i^T = [c_{i1} \ \ 0] \ , \tag{5.5}$$

to reflect the fact that we penalize only z_i's in the performance index. Equation (5.2), together with the output equation

$$y = Cx \ , \tag{5.6}$$

describes the system S in (3.1).

We now apply to S the model reduction scheme based on the balanced representation of S. However, before doing this we note from Table 1 that the first three modes are undamped, i.e., the blocks A_1, A_2 and A_3 in (5.3) are not stable. This prevents the model reduction scheme from being applied directly, for it requires the system be stable. To avoid this difficulty, we partition the state vector as $x = [x_u^T \ x_d^T]^T$, where $x_u = [x_1^T \ x_2^T \ x_3^T]^T$ and $x_d = [x_4^T \ x_5^T \ \ldots \ x_9^T]^T$ correspond respectively to the undamped and damped modes. Partitioning A and B of (5.2) and C of (5.6) accordingly, we represent S as the parallel connection of two systems S_u and S_d, which are described by

$$\begin{aligned} S_u\colon \dot{x}_u &= A_u x_u + B_u u \\ y_u &= C_u x_u \ , \end{aligned} \tag{5.7}$$

$$\begin{aligned} S_d\colon \dot{x}_d &= A_d x_d + B_d u \\ y_d &= C_d x_d \end{aligned} \tag{5.8}$$

Now since S_d is stable, we can obtain a reduced order model $\bar{S}_d$ for S_d based on its balanced representation. The parallel connection of S_u and $\bar{S}_d$ then gives a reduced order model $\bar{S}$ of S. In other words, we keep the undamped modes in the reduced order model and add to these a part of the damped modes.

We choose $R = 10^{-5}I$, and c_{i1} in (5.5) as

$$c_{i1} = \begin{cases} 1 & i = 1, 2, 3 \\ 0.1 \ , & i = 4, 5, \ldots, 9 \end{cases}$$

to penalize the undamped modes more than the damped ones. The optimal gain matrix and the eigenvalues of the optimal system are shown in Table 2, along with the singular values of the product $\Sigma_0 \Sigma_C$ of the observability and controllability grammians of S_d. From these values we observe that the last four singular values are very small compared to the first eight, which suggests that an 8-th order reduced model should adequately represent S_d. The results of suboptimal design based on an 8-th order reduced model of S_d are shown in Table 3. The suboptimal gains and eigenvalues of the suboptimal system coincide with those of the optimal ones up to the fourth decimal place, producing a degree of suboptimality $\mu = 1.000$ and thus confirming our observation. The results of the suboptimal design based on a second order reduced model of S_d are shown in Table 4, which shows a 67% loss of performance quality corresponding to an 83% reduction in the order of S_d.

4.6 CONCLUSIONS

All model-reduction schemes available in the literature are concerned with the open-loop behavior of systems, and therefore, lack the justification of producing lower order models which would also represent adequately the closed-loop performance of the systems they approximate. The suboptimality concept outlined in this paper provides a means for validation of reduced-order models by comparing the performance of closed-loop reduced model with that of the full-order system. The advantages of this validation approach are:

(i) It is particularly suitable for measuring effectiveness of controls under information structure constraints;

(ii) It can be successfully applied to interconnected systems to produce suboptimal control;

(iii) It provides robustness measures such as gain and phase margin, explicitly in terms of the degree of suboptimality.

A drawback of the suboptimality approach is the necessity to solve the control problem for the full-order system to be used as reference. This may not be very important unless the order of the system is extremely large or it contains uncertian parameters, but becomes a challenging difficulty in these cases.

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Table 1. Parameters of the model of the large space structure

i	d_i (x 10^{-2})	ω_i	b_i^T (x 10^{-4})	
1	0.	0.	3.460	3.460
2	0.	0.148	-9.120	34.400
3	0.	0.288	-1.300	0.548
4	1.414	1.414	-1.400	36.000
5	1.829	1.829	-4.070	-34.700
6	2.846	2.846	28.500	39.500
7	3.831	3.831	71.800	-24.600
8	8.798	8.798	-8.000 x 10^{-4}	-1.150 x 10^{-5}
9	9.613	9.613	-0.172	-4.130 x 10^{-3}

Table 2. Optimal Design and Singular Values of $\Sigma_0\Sigma_C$

OPTIMAL GAIN MATRIX (COLUMNS 1 TO 6)

0.2998E 03	0.1398E 04	-0.9208E 02	-0.2136E 03	-0.2960E 03	0.3121E 03
0.1005E 03	0.3892E 03	0.2953E 03	0.3899E 03	-0.6539E 02	0.2092E 02

OPTIMAL GAIN MATRIX (COLUMNS 6 TO 12)

0.3381E 01	0.2121E 01	0.1080E 01	-0.2634E 01	0.3069E 01	0.3647E 01
0.2261E 02	0.8127E 01	-0.1787E 02	-0.7468E 01	-0.9151E 01	-0.4424E 01

OPTIMAL GAIN MATRIX (COLUMNS 12 TO 18)

0.4380E 01	0.5322E 01	-0.4572E-05	-0.5848E-05	-0.7579E-03	-0.9646E-03
0.4559E 01	-0.1416E 01	0.1311E-05	-0.1068E-06	0.2002E-03	-0.2618E-04

OPTIMAL EIGENVALUES

RE	IM
-0.4907E-01	0.9613E 01
-0.4807E-01	-0.9613E 01
-0.4399E-01	0.8798E 01
-0.4399E-01	-0.8798E 01
-0.4132E-01	0.3268E 01
-0.4132E-01	-0.3268E 01
-0.3029E-01	0.2845E 01
-0.3020E-01	-0.2845E 01
-0.3014E-01	0.1829E 01
-0.3014E-01	-0.1829E 01
-0.3537E-01	0.1414E 01
-0.3537E-01	-0.1414E 01
-0.3215E-01	0.2843E 00
-0.3215E-01	-0.2643E 00
-0.7430E 00	0.7585E 00
-0.7430E 00	-0.7565E 00
-0.2568E 00	0.2629E 00
-0.2568E 00	-0.2629E 00

SINGULAR VALUES OF $W_O W_C$

0.1364E-03	0.1127E-03	0.4021E-04	0.1817E-04	0.9167E-05	0.8128E-05
0.7243E-05	0.5585E-05	0.8753E-12	0.8578E-12	0.2664E-16	0.2610E-16

Table 3. Suboptimal Design I

ORDER OF THE REDUCED SYSTEM = 8

SUBOPTIMAL GAIN MATRIX (COLUMNS 1 TO 6)

0.2998E 03	0.1398E 04	-0.9208E 02	-0.2136E 03	-0.2960E 03	0.3121E 03
0.1005E 03	0.3892E 03	0.2953E 03	0.3899E 03	-0.6539E 02	0.2092E 02

SUBOPTIMAL GAIN MATRIX (COLUMNS 6 TO 12)

0.3381E 01	0.2121E 01	0.1080E 01	-0.2634E 01	0.3069E 01	0.3647E 01
0.2261E 02	0.8127E 01	-0.1787E 02	-0.7468E 01	-0.9151E 01	-0.4424E 01

SUBOPTIMAL GAIN MATRIX (COLUMNS 12 TO 18)

0.4580E 01	0.5322E 01	0.1637E-03	0.2118E-04	0.7986E-06	-0.4600E-07
0.4539E 01	-0.1416E 01	-0.1162E-04	-0.4589E-04	0.3365E-06	0.1004E-06

SUBOPTIMAL EIGENVALUES

RE	IM
-0.4807E-01	0.9613E 01
-0.4807E-01	-0.9613E 01
-0.4399E-01	0.8798E 01
-0.4399E-01	-0.8798E 01
-0.4132E-01	0.3268E 01
-0.4132E-01	-0.3268E 01
-0.3030E-01	0.2846E 01
-0.3030E-01	-0.2846E 01
-0.3014E-01	0.1829E 01
-0.3014E-01	-0.1829E 01
-0.3597E-01	0.1414E 01
-0.3597E-01	-0.1414E 01
-0.3215E-01	0.2843E 00
-0.3215E-01	-0.2843E 00
-0.7430E 00	0.7585E 00
-0.7430E 00	-0.7585E 00
-0.2568E 00	0.2629E 00
-0.2568E 00	-0.2629E 00

DEGREE OF SUBOPTIMALITY = 0.1000E 01

Table 4. Suboptimal Design II

ORDER OF THE REDUCED SYSTEM = 2

SUBOPTIMAL GAIN MATRIX (COLUMNS 1 TO 6)

0.3005E 03	0.1400E 04	-0.9093E 02	-0.2090E 03	-0.2964E 03	0.3133E 03
0.9836E 02	0.3764E 03	0.2963E 03	0.3783E 03	-0.6326E 02	0.2004E 02

SUBOPTIMAL GAIN MATRIX (COLUMNS 6 TO 12)

0.7902E-01	0.9640E-01	-0.7305E-02	0.3679E-01	-0.5248E-02	-0.3086E-01
0.4662E-01	-0.8469E-02	0.1237E-02	0.2041E-01	-0.2614E-02	-0.1764E-01

SUBOPTIMAL GAIN MATRIX (COLUMNS 12 TO 18)

0.1034E 01	0.4096E-01	0.3470E-04	-0.6722E-06	0.1853E-06	-0.1055E-07
0.6037E 00	0.5238E-02	0.2045E-04	-0.4006E-06	0.1093E-06	-0.6106E-08

SUBOPTIMAL EIGENVALUES

RE	IM
-0.4807E-01	0.9613E 01
-0.4807E-01	-0.9613E 01
-0.4399E-01	0.8798E 01
-0.4399E-01	-0.8798E 01
-0.1940E-01	0.3249E 01
-0.1940E-01	-0.3269E 01
-0.1422E-01	0.2846E 01
-0.1422E-01	-0.2846E 01
-0.9109E-02	0.1829E 01
-0.9109E-02	-0.1829E 01
-0.7048E-02	0.1414E 01
-0.7048E-02	-0.1414E 01
-0.7441E 00	0.7563E 00
-0.7441E 00	-0.7583E 00
-0.2568E 00	0.2628E 00
-0.2568E 00	-0.2628E 00
-0.3250E-01	0.2843E 00
-0.3250E-01	-0.2843E 00

DEGREE OF SUBOPTIMALITY = 0.3317E 00

5.0 PARAMETER PLANE

The utilization of the parameter plane theory provides the control system designer with an additional degree of freedom to be utilized in designing control systems. In a sense it is a two demensional root-locus in that one may design the control system through pole placement techniques. The Control Dynamics Company's standard practice is to design control systems utilizing the power of the parameter plane. Proper use of the information will provide the control system designer with not only expected performance at the operating point but one can also obtain a feel of the relative "robustness" of the design if the selected point is comfortably away from the stability boundaries.

We will present first (Section 5.1) an overview of the theory of the parameter plane from the standpoint of a continuous system. Then, in Section 5.2 the extension of this fundamental theory to include systems of higher order will be covered. Next, (Section 5.3) we will disucss an overview of the extension of the continuous domain theory to include digital systems. Finally in Section 5.4 we will present some typical results from a digital application.

5.1 OVERVIEW

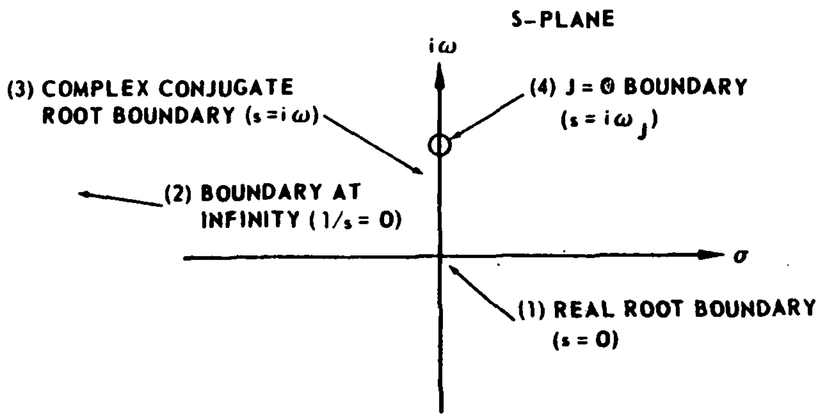
We begin our study of the theory of the parameter plane by investigating the specification of the stability boundaries. We see that there are three boundaries present, just as there are in the root locus theory. There is a boundary at $s = 0$, a boundary at ∞ and the complex conjugate boundary at $s = iw$ (the imaginary axis in the root locus). One represents the characteristic equation in terms of two arbitrary constants, each of which multiplies a polonominal in s, and the remainder of the characteristic equation. These stability boundaries can then be represented in terms of these two constants and plotted as a function of these two constants, hence the parameter plane. This is summarized on Fig. 5.1.

Now that one has the solution of the two equations arising from setting the real and imaginary parts of the characteristic equation to zero, one is made aware of another (fourth) stability boundary. This boundary occurs if the Jacobian is equal to zero. In addition the sign of the Jacobian provides one information on how the roots of the characteristic equation behave as the particular boundary is crossed. (See Fig. 5.2.) Once the stability boundaries are established, then one is concerned about the performance of the system. This performance can be determined or specified by plotting the contours of closed loop damping and frequency on the same parameter plane. One first must substitute for the real and complex parts of the variables, as shown in Fig. 5.3.

Fig. 5.1. **D – DECOMPOSITION**

$$\Delta(s) = \sum_{p=o}^{n} f_p s^p = 0 \qquad f_p = a_p x + b_p y + c_p$$

TO MAP: STABILITY BOUNDARIES FROM S–PLANE TO x,y PLANE



(1) REAL ROOT BOUNDARY: $s = 0$

x,y MAP: $f_o = a_o x + b_o y + c_o = 0$

(2) BOUNDARY AT INFINITY: $1/s = 0$

x,y MAP: $f_n = a_n x + b_n y + c_n = 0$

(3) COMPLEX CONJUGATE ROOT BOUNDARIES: $s = i\omega$

$$Re\left\{\Delta(i\omega)\right\} = f_o - \omega^2 f_2 + \omega^4 f_4 - \cdots = 0$$

$$Im\left\{\Delta(i\omega)\right\} = i\omega\left[f_1 - \omega^2 f_3 + \omega^5 f_5 - \cdots\right] = 0$$

REARRANGE:

$$Re\left\{\Delta(i\omega)\right\} = A_1(\omega^2)x + B_1(\omega^2)y + C_1(\omega^2) = 0$$

$$Im\left\{\Delta(i\omega)\right\} = A_2(\omega^2)x + B_2(\omega^2)y + C_2(\omega^2) = 0$$

SOLVE FOR x,y:

Fig. 5.2. **D – DECOMPOSITION (CONT'D)**

$$x = \frac{B_1 C_2 - B_2 C_1}{J} \qquad y = \frac{A_2 C_1 - A_1 C_2}{J}$$

JACOBIAN: $J = A_1 B_2 - A_2 B_1$

(4) J=0 BOUNDARY (HISTORICALLY OVERLOOKED)
WHEN Re & Im EQUATIONS IDENTICAL

SHADING RULES

TO PORTRAY HOW ROOTS OF $\Delta(s)$ CROSS BOUNDARIES
IN x–y PLANE; USUALLY CAN DEDUCE STABLE REGION

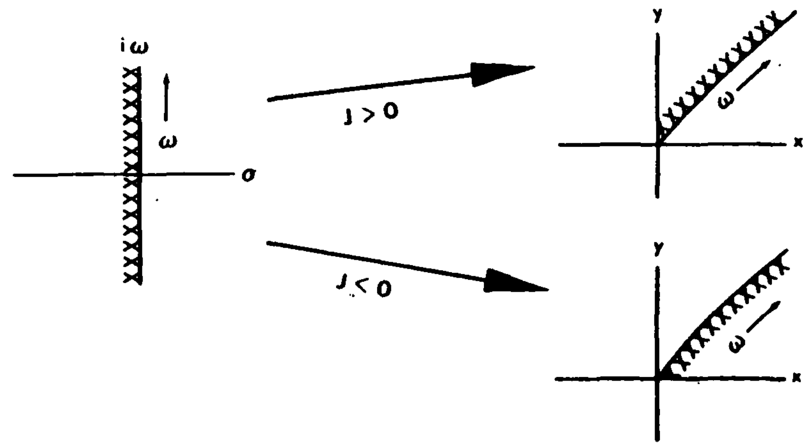
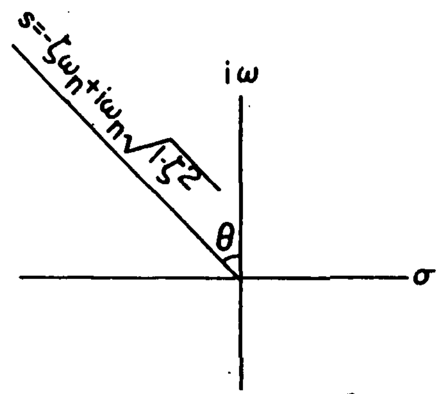


Fig. 5.3. PARAMETER PLANE

MAP CHARACTERISTICS CONTOURS FROM s-PLANE ONTO x,y PLANE



$$\omega_n = \frac{\omega}{\sqrt{1-\zeta^2}}$$

$$\zeta = \sin^{-1}\theta$$

SUBSTITUTE $S^p = (-\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2}\,)^p$ INTO CHARACTERISTIC

EQUATION:

$$\Delta(s) = \sum_{p=0}^{n} f_p s^p = 0$$

SIMPLIFY PROBLEM:

$$s^p = X_p + iY_p$$

$$X_p = 2\zeta\omega_n X_{p-1} - \omega_n^2 X_{p-2}$$

$$Y_p = 2\zeta\omega_n Y_{p-1} - \omega_n^2 Y_{p-2}$$

$$s^0 = X_0 + iY_0 = 1 + i0$$

$$s^1 = X_1 + iY_1 = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2}$$

Now we must go through exactly the same procedure that we did to determine the stability boundaries. That is, we must equate the real and imaginary parts to zero. Thereby we obtain two equations in two unknowns that can be solved. The solution of these equations then has a similar form to the one that we experienced before. (See Fig. 5.4.).

Of course, one must allow for the occurence of real roots in the stable region. One assumes a form of the real root and makes substitution into the characteristic equation. The result is an equation in two unknowns that can be solved parametrically and then plotted on the same parameter plane. (See Fig. 5.5.).

Thus we have all the tools necessary to completely describe the roots of the governing characteristic equation, and thus the system response, in terms of two unknown parameters. We now can select these two parameters for the particular system to achieve the desired stability and response characteristics. We will wait until we have completed the discussion of the digital parameter plane before presenting any numerical examples of the technique.

Fig. 5.4 PARAMETER PLANE (CONT'D)

CHARACTERISTIC CONTOURS IN X-Y PARAMETER PLANE:

$$\Delta(s) = \sum_{p=o}^{n} f_p (X_p + i Y_p)$$

$$= \sum_{p=o}^{n} (a_p x + b_p y + c_p) X_p + i \sum_{p=o}^{n} (a_p x + b_p y + c_p) Y_p = 0$$

$$\mathrm{Re}\{\Delta(s)\} = A_1(\zeta, \omega_n) x + B_1(\zeta, \omega_n) y + C_1(\zeta, \omega_n) = 0$$

$$\mathrm{Im}\{\Delta(s)\} = A_2(\zeta, \omega_n) x + B_2(\zeta, \omega_n) y + C_2(\zeta, \omega_n) = 0$$

$$A_1 = \sum_{p=o}^{n} a_p X_p \quad B_1 = \sum_{p=o}^{n} b_p X_p \quad C_1 = \sum_{p=o}^{n} c_p X_p$$

$$A_2 = \sum_{p=o}^{n} a_p Y_p \quad B_2 = \sum_{p=o}^{n} b_p Y_p \quad C_2 = \sum_{p=o}^{n} c_p Y_p$$

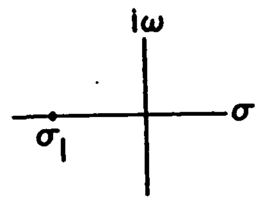
$$x = \frac{B_1 C_2 - B_2 C_1}{J} \qquad y = \frac{A_2 C_1 - A_1 C_2}{J}$$

$$J = A_1 B_2 - A_2 B_1$$

Fig. 5.5

PARAMETER PLANE (CONT'D)

REAL ROOT CONTOURS



LET $s=\sigma_1$ IN $\Delta(s)$

$$A_3 x + B_3 y + C_3 = 0$$

$$A_3 = \sum_{p=0}^{n} a_p \sigma_1^{\,p} \qquad B_3 = \sum_{p=0}^{n} b_p \sigma_1^{\,p} \qquad C_3 = \sum_{p=0}^{n} c_p \sigma_1^{\,p}$$

5.2 MATRIX FORMULATION EXTENSION

5.2.1 INTRODUCTION

In this section we focus our attention on the mathematical model

$$M\ddot{\eta} + C\dot{\eta} + K\eta = f \tag{1}$$

where M,C, and K are nxn matrices, and η and f are n-dimensional vectors. The feedback f is assumed to be of the form

$$f = \Phi F \Theta \tag{2}$$

where $\Phi = \{\phi_1, \phi_2\}$ is composed of two n-vectors, and has the "PID" form

$$\Theta = \begin{bmatrix} (\phi_1, \dot{\eta}) \\ \int (\phi_1, \dot{\eta})dt \\ \iint (\phi_1, \dot{\eta})dt_1\, dt_2 \\ (\phi_2, \dot{\eta}) \\ \int (\phi_2, \dot{\eta})dt \\ \iint (\phi_2, \dot{\eta})dt_1\, dt_2 \end{bmatrix} \tag{3}$$

and $F = (f_{ij})$ is a 2x6 feedback matrix of constant gains. The notation (,) stands for the inner product of two vectors.

The purpose of this portion of the report is to find a convenient method for performing a parameter plane analysis when the parameters of interest are any two elements of the F matrix. Thus, we implicitly assume that all of the pertinent elements in the above three equations are numerically know from a separate design analysis. One benefit of the parameter plane technique would then be a measure of robustness for the "design point".

The analysis that follows does not assume any special form of the matrices other than that previously described. Thus, this technique is itself robust as it can be applied to any model which can be described by the equations. The FORTRAN program developed for this purpose is included under the name DARPA. It is very interactive and only requires the user to specify the two elements in the F matrix and the range of values of the complex number which is a third parameter in the parameter plane technique. Since the particular form of the mathematical model is almost diagonal, a simple check case can be performed by assuming the F matrix to be identically zero. Of course, this would not be the case in practice, but was used to verify the results of the more general theory. This simplified version is included as the FORTRAN program LDARPA (Little DARPA).

A second use of the parameter plane technique is to generate two elements in the feedback matrix F that provide a "design point". If contours are plotted for a fixed damping factor and if one selects a point of crossing in the stable region, then this design point will furnish a system that is stable and at least two modes will have a known damping factor and frequency.

An attempt is then made to focus attention on two parameters in the vector. These generalized locations and slopes are thought to be critical and variations about the assumed values are deemed important. Initial efforts are centered on the first four components in these vectors, and (as later discovered) due to the degenerate form of the sample mathematical model, no positive results

can be obtained. Lack of time prevented further analysis on this portion of the research but it is felt that this type of analysis can still prove to be fruitful if damping is included in the third and/or fourth modes or if components greater then four are investigated.

5.2.2 MATHEMATICAL ANALYSIS

The mathematical analysis that follows assumes the two elements of interest are elements of the F matrix. These two elements will be denoted as (α , β) will be allowed to vary, and the remaining elements in Eqs. 1-3 will be assumed fixed. The central concept of the parameter plane technique is to discover those values of (α,β) which, for a fixed complex number, satisfy the characteristic equation of the system. However, the derivation of the characteristic equation is a formidable task and the alternate technique of evaluating the determinant of the Laplace transform of the system equations will yield identical results. Since the parameters (α,β) are spread throughout the equations, one must first determine the generic (albeit theoretical) form of the characteristic equation. For this purpose we decompose F as follows:

$$F = \left[\begin{array}{ccc|ccc} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ \hline f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \end{array}\right] \qquad (4a)$$

$$\underline{\Delta} = \left[\begin{array}{c|c} \text{Bank 1} & \text{Bank 2} \\ \hline \text{Bank 3} & \text{Bank 4} \end{array}\right] \tag{4b}$$

Thus, to say that α is in Bank 4 will mean that $\alpha = f_{24}$, or f_{25}, or f_{26}.

DEFINITION: The elements α and β are in diagonal banks if;

α is an element of Bank 1 and β is an element of Bank 4,

or β is an element of Bank 1 and α is an element of Bank 4,

or α is an element of Bank 2 and β is an element of Bank 3,

or β is an element of Bank 2 and α is an element of Bank 3.

Otherwise, α and β are in nondiagonal banks.

The generic form of the characteristic equation assumes one of two forms and the choice of the form is dependent on the location of α and β. More specifically, we have the following:

THEORM (i) If (α,β) are in nondiagonal banks, then the characteristic equation of Eqs. 1-3 is equivalent to

$$\Delta(s) = 1 + \alpha b(s) + \beta c(s) = 0, \tag{5}$$

where $b(s)$ and $c(s)$ are rational polynomials independent of (α,β).

(ii) If (α,β) are in diagonal banks, then the characteristic equation is equivalent to

$$\Delta(s) = 1 + \alpha b(s) + \beta c(s) + \alpha\beta d(s) \tag{6}$$

where $b(s)$, $c(s)$, and $d(s)$ are rational polynomials independent of (α,β).

REMARK. The parameter plane loci is now determined by selecting convenient values of (α,β) to evaluate the left side of Eq. 5 or

Eq. 6. The information is then used to find b(s), c(s), and/or d(s), from which (α,β) is determined by solving Eq. 5 or Eq. 6. This portion of the analysis is continued in the next section. The reader not interested in the detailed mathematical analysis of proving the above theorem may wish to skip to that section.

The proof of the theorem depends on the following well known lemma:

LEMMA 1. Let I denote the identity matrix and b,c, be nonzero vectors. Then

$$\det(I - bc^T) = 1 - (c,b) \tag{7}$$

PROOF. Since $(I - bc^T)b = \{1 - (c,b)\}b$, we have that b is an eigenvector of $I - bc^T$ with eigenvalue $1 - (c,b)$. Let x be any other eigenvector with corresponding eigenvalue s, so that $(I - bc^T)x = sx$ implies $(1 - s)x = (c,x)b$. Thus, if s is not equal to 1 then x may be selected as b above. If s = 1, then the equation $(c,x) = 0$ has n-1 independent solutions. Thus, the spectrum of $I - bc^T$ consists of the two eigenvalues $1 - (c,b)$ and 1. The proof of the lemma is completed by noting that the product of the eigenvalues of a matrix is equal to the determinant of the matrix.

We also require the following extension of Lemma 1.

LEMMA 2. Let

$$C = I - bc^T - de^T$$

and assume the vectors b and d to be independent as well as c and e. Further assume that $(c,d) \neq 0$ and $(e,b) \neq 0$. Then

$$\det(C) = 1 - (c,b) - (e,d) + (c,b)(e,d) - (c,d)(e,d). \tag{8}$$

PROOF. As in the proof of Lemma 1, we show that the spectrum of C is composed of the three eigenvalues $\{1, s_1, s_2\}$ where

$$s_j = 1 - (c,b) - u_j(c,d), \quad j = 1,2$$

and u_j are roots of the quadratic equation.

$$(c,d)u^2 + \{(c,d) - (e,d)\}u - (e,b) = 0. \tag{9}$$

For this purpose, we observe that there are n-2 independent eigenvectors with eigenvalues equal to one. That is, $Cx = x$ yields

$$x - b(c,x) - d(e,x) = x \quad \text{or}$$
$$b(c,x) + d(e,x) = 0.$$

Since b and d are independnet vectors, we conclude from this last equation that $(c,x) = 0$ and $(e,x) = 0$. The assumed independence of c and e implies that there are n-2 distinct solutions to these two last equations.

Now let $x = b + ud$ so that $Cx = sx$ and the independence of b and d yield

$$1 - (c,d) - u(c,d) = s, \text{ and}$$
$$u - (e,b) - u(e,d) = su.$$

If we multiply the first equation by -u and add to the second equation, we obtain Eq. 9. Now the roots of Eq. 9 satisfy

$$-(c,d)(u_1 + u_2) = (c,b) - (e,d), \text{ and}$$
$$(c,d)u_1u_2 = -(e,b).$$

Thus,

$$\det(C) = s_1s_2 = [\{1 - (c,b)\} - u_1(c,d)][\{1 - (c,b)\} - u_2(c,d)],$$

and Eq. 8 follows by expanding this last result.

REMARK. The assumption of the vectors being nonorthogonal can be relaxed, but this is not required in this report.

We now return to the proof of the Theorem.

PROOF. (i) To solidify ideas let $\alpha = f_{11}$ and $\beta = f_{12}$ so that (α, β) are in Bank 1. Then

$$F = F' + \begin{bmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the 2x6 F' matrix is formed from F by deletions α and β .

Thus,

$$F\Theta = F'\Theta + \{\alpha(\phi_1, \dot{\eta}) + \beta\int(\phi_1, \eta)dt \quad 0\}^T$$

and

$$f = \Phi F'\Theta + \alpha\phi_1(\phi_1, \dot{\eta}) + \beta\phi_1\int(\phi_1, \dot{\eta})dt$$

To simplify notation we define the nxn matrix R(s) as follows:

$$R(s) = Ms^2 + Cs + K - \Phi F' \{s\phi_1,\ \phi_1,\ s^{-1}\phi_1,\quad \phi_2,\ \phi_2,\ s^{-1}\phi_2\}.$$

Then the Laplace transform of Eqs. 1-3 yields.

$$\{R(s) - (\alpha s + \beta)\ \phi_1\ \phi_1^T\}\eta\ (s) = I.V. \qquad (10)$$

where $\eta(s)$ is the Laplace transform of $\eta(t)$ and I.V. stands for initial values. Thus, the characteristic polynomial of the system becomes

$$p(s) = \det \{R(s) - (\alpha s + \beta)\ \phi_1\phi_1^T\}.$$

If we assume the inverse of R(s) exists, then we can factor out R(s) on the left to obtain the identity

$$R(s) - (\alpha s + \beta)\phi_1\phi_1^T = R(s)\ \{I - (\alpha s + \beta)R^{-1}(s)\phi_1\phi_1^T\}.$$

Using Lemma 1 with $b = (\alpha s + \beta)R^{-1}(s)\phi_1$, $c = \phi_1$, and using the last identity yields the following form for the characteristic polynomial

$$p(s) = \det\{R(s)\}\{1 - (\alpha s + \beta)(\phi_1, R^{-1}(s)\phi_1)\}.$$

Thus, $p(s) = 0$ implies the characteristic equation can be expressed as

$$\Delta(s) = 1 - \{s(\phi_1, R^{-1}(s)\phi_1)\}\alpha - (\phi_1, R^{-1}(s)\phi_1)\beta.$$

In general, the inverse of R(s) exists as it is composed of rational polynomials, and the theorem is proved for the special case where $c(s) = -(\phi_1, R^{-1}(s)\phi_1)$ and $b(s) = sc(s)$. The remaining parts of (i) can be proved by moving the elements α and β to their specified locations and repeating this proof.

(ii) Again to solidify ideas, we let $\alpha = f_{11}$ be in Bank 1 and $\beta = f_{24}$ be in Bank 4. Then

$$f = \Phi F'\Theta + \Phi \begin{bmatrix} (\phi_1, \dot{n})\alpha \\ (\phi_2\ \dot{n})\beta \end{bmatrix}$$

where F' is formed from F by deletion α and β. Thus

$$p(s) = \det\{R(s) - \alpha s\phi_1\phi_1^T - \beta s\phi_2\phi_2^T\}.$$

Using Lemma 2 with $b = -\alpha sR^{-1}\phi_1$, $c = \phi_1$, $d = -\beta sR^{-1}\phi_2$, and $e = \phi_2$ yields

$$p(s) = \det\{R(s)\}\{1 - \alpha s(\phi_1, R^{-1}\phi_1) - \beta s(\phi_2, R^{-1}\phi_2)$$
$$+ \alpha\beta[(\phi_1, R^{-1}\phi_1)(\phi_2, R^{-1}\phi_2) - (\phi_1, R^{-1}\phi_2)(\phi_2, R^{-1}\phi_1)]\}s^2$$

so that

$$b(s) = -s(\phi_1, R^{-1}\phi_1),$$

$$c(s) = -s(\phi_2, R^{-1}\phi_2), \text{ and}$$

$$d(s) = s^2\{(\phi_1, R^{-1}\phi_1)(\phi_2, R^{-1}\phi_2) - (\phi_1, R^{-1}\phi_2)(\phi_2, R^{-1}\phi_1)\}.$$

5.2.3 COMPUTER PROGRAMS

In general, the R matrix as defined by Eq. 10 will not be sparse and the evaluation of b(s), c(s), and d(s) in the Theorem will require considerable computational effort to determine its inverse. To alleviate computing this inverse for each value of s, we return to the original form of the characteristic polynomial. That is,

$$p(s;\alpha,\beta) = \det\{Ms^2 + Cs + K - \Phi F(\alpha,\beta)\Theta(s)\}. \quad (11)$$

Then using Eq. 11 with Eq. 5 (or Eq. 6) yields

$$\begin{aligned} p(s;0,0) &= \det\{R(s)\} \\ p(s;1,0) &= \det\{R(s)\}\{1 + b(s)\} \\ p(s;0,1) &= \det\{R(s)\}\{1 + c(s)\} \end{aligned} \quad (12)$$

(and for Eq. 6: $p(s;1,1) = \det\{R(s)\}\{1 + b(s) + c(s) + d(s)\}$).
The information on the left of Eq. 12 is found by numerically evaluating the complex determinants by using a modified Gaussian elimination scheme to handle complex elements. As Eq. 5 or Eq. 6 gives the explicit form of the characteristic equation, we can now solve for b(s), c(s), and, if needed, d(s). Finally, the values of α and β that satisfy the characteristic equation are found by taking the real and imaginery parts of Eq. 5 or Eq. 6 and solving the resulting equations. For Eq. 5, the resulting equations are linear in the two unknowns while for Eq. 6 the equations are nonlinear and require a more delicate routine for their solutions. These subroutines have been denoted as SOLVE1 and SOLVE2, respectively.

The analysis of the problem has been coded in FORTRAN using double precision arithmetic. Extensive use is made of complex arithmetic, but for the sake of program portability no use is made of FORTRAN complex variables. The main line program is name DARPA and follows the presentation of the previous paragraph. DARPA requires the initial matrices M,C,K,Φ, and F from the user's data base and then requests for the row and column of the desired elements in the F matrix that are to be used in the parameter plane analysis. Then the user is prompted for a range of values of the variable s. DARPA calls subroutines BUILD, MMULTD, CDIV, COND, CGAUSS, SOLVE1, AND SOLVE2. Subroutine BUILD is used to construct the real and imaginary parts of the matrix in Eq. 11 with F replaced by F'. (The present version of BUILD assumes the special form of the mathematical model for test purposes.) MMULTD is a double precision multiplication routine, DCIV is a double precision complex variable division routine, CGAUSS is a modification of Gaussian elimination to handle complex numbers. For some values of s, round off-error may be experienced, so a conditioning routine COND is included to scale the elements of the matrices before calling GAUSS. Finally, SOLVE1 and SOLVE2 as mentioned in the previous paragraph are used to solve the resulting equations.

A sample run of DARPA is included after the listings of the program. The first part of the run lists the current numerical values of the input matrices (F was assumed to be zero for the test case). The second part of the output prompts the user for

the values of α and β. In the example $\alpha = f_{11}$ and $\beta = f_{12}$ were selected as the parameters of interest. Then the user is prompted for initial values of s = SIG + j OMEGA, incremental values of s and final values of s. In the example at hand two points s = .1j and s - .2j were run. Note in the example that the increment in SIG was zero so a nonzero value was selected for the final value of SIG. The results of the calculations are given next. In order to determine some accuracy of results, the calculated values were substituted back into the original equation and the determinant was evaluated once again. This determinant should be zero. The final calculation shows this diagonal elements of this determinant after calling CGAUSS. As the determinant will be the product of these diagonal elements, it is seen that the results are within predictable accuracy (note the last two elements). To gain further confidence in the DARPA code a separate and independent analysis was performed on the specific form of the equations. This is described in the next few paragraphs, and by comparing the results of DARPA and LDARPA we note the excellent agreement of the numerical results.

If we assume $F' = 0$ then the R(s) matrix above Eq. 10 is

$$R(s) = Ms^2 + Cs + K.$$

As R(s) for the test case is almost diagonal an explicit expression for its inverse is easy to obtain and one can obtain a much simplier form of the characteristic equation of the closed loop system. For example, if $\alpha = f_{11}$ and $\beta = f_{12}$ we have already seen that

$$\Delta(s) = 1 - \alpha\{s(\phi_1, R^{-1}(s)\phi_1)\} - \beta(\phi_1, R^{-1}(s)\phi_1).$$

Substituting $R^{-1}(s)$ in this last expression and expanding the inner products gives us the following expression for the characteristic equation:

$$\Delta(s) = 1 - (\alpha s + \beta)\ \{\sum_{j=3}^{10}\phi_1^2(j) / \left[s^2 + c(j)s + k(j)\right] \tag{13}$$

$$+ \left[m(1,1) + m(2,2) - 2m(1,2)\right] / \left[s^2\ (m(1,1)m(2,2) - m(1,2)^2)\right]\ \}$$

where c(j) and k(j) are the diagonal elements of C and K, respectively. This formulation allows for a much smaller program with much less chance for numerical roundoff. As mentioned above the coding of Eq. 13 is called LDARPA.

5.2.4 DISCUSSION

Fig. 1 is a sketch of the parameter plane with $\sigma = o$. The parameters selected were $\alpha = f_{11}$ and $\beta = f_{12}$. The three pages with the title PRODUCTION RUN FOR SIGMA = 0 are the actual numerical values used in producing this sketch and were generated using LDARPA. Note that this sketch is not to scale due to the large dispersion of the magnitude of the numerical values. This sketch separates the plane into stable and unstable regions. The static boundary (the f_{11} - axis) was determined by setting $s = 0$ and performing a separate analysis. The stable region was determined by reducing the problem to one of smaller dimension and invoking the Routh-Hurwitz stability test. For all intents the stable region consists of the third quadrant. Fig. 2 is an enlargement

of the region close to the origin. It merely shows that for small positive values of f the system will remain stable provided f_{12} is negative.

Also included under the title PARTIAL PRODUCTION RUN FOR SIGMA = -.01414 are some numerical values for the parameter plane with σ fixed and equal to the value in the title. Lack of time and space prohibits exhibiting the graph of this data and the other numerious numerical values generated. However, it will be noted that a careful scan was made of the data for σ = -.707 and no multiple crossing of the graphs were detected.

Some attempt was made to perform a parameter plane on the elements in the Φ matrix. For a full F matrix the resulting equations appear to be highly nonlinear if two arbitray values are selected, hence no numerical experiments were performed. However, for the special model considered, and if only two values of the F matrix are nonzero then one can still get some partial results. For this purpose, let us assume f_{11} and f_{12} are the nonzero elements and and concentrate on the first bending mode. That is let

$$x = \phi_1(3) \qquad \text{and} \qquad y = \phi_2(4),$$

so that Eq. 13 can be rewritten as

$$\frac{x^2}{s^2 + c(3)s + k(3)} + \frac{y^2}{s^2 = c(4)s + k(4)} + \frac{1}{(f_{11}s + f_{12})} - \frac{m(1,1) + m(2,2) - 2m(1,2)}{m(1,1)m(2,2) - m(1,2)^2} - \sum_{j=5}^{10} \frac{\phi_i^2(j)}{s^2 + c(j)s + k(j)} .$$

From the given numerical data $c(3) = c(4) = 0$. If one now substitutes $s = j\omega$ in this last expression we run into a contradiction as the imaginary part of the right hand side must be zero. To alleviate this problem one must assign appropriate values to the modal damping matrix C.

```
      DOUBLE PRECISION MASS(4),C(6),STIFF(8),PHI1(10),PHI2(10)
      DOUBLE PRECISION F(2,6),X(10,10),Y(10,10),WORK(10,10),U(6,10)
      DOUBLE PRECISION SIG,W,DSIG,DW,FSIG,FW,AR,AI,BR,BI,CR,CI,DR,DI
      DOUBLE PRECISION ALPHA,BETA,ALPHA1,ALPHA2,BETA1,BETA2
      DOUBLE PRECISION MAXA,MAXB,MAXC,MAXD,RATIO
      DIMENSION IPVT(10)
      READ(6,100)C
      READ(6,100)STIFF
      READ(6,100)PHI1
      READ(6,100)PHI2
      READ(6,100)MASS
      N = 10
      DO 4 I = 1,N
      DO 4 J = 1,N
      F(I,J) = 0.0D0
 4    CONTINUE
C...CURRENTLY ASSUMING F IS ZERO.  IF NOT, UPDATE DATABASE AND INSERT A
C...READ(6,100) F.   NOTE THAT MASS,C,AND STIFF ARE VERY SPECIAL AND
C...ONLY INCLUDE THE NONTRIVIAL ELEMENTS.  IF THEY WERE GENERAL THEN
C...WILL NEED TO CHANGE THE INPUT OUTPUT PORTIONS OF THE PROGRAM.
C  FOLLOWING OUTPUT DATA FORMAT IS FOR THIS SPECIAL CASE
100   FORMAT(10D11.6)
      WRITE(5,200)MASS
200   FORMAT(' M11,M12,M21,M22 = ',/,4D12.6)
      WRITE(5,201)C
201   FORMAT(' LAST 6 ELEMENTS OF DIAG. DAMPING MATRIX = ',/,6D12.6)
      WRITE(5,202)
202   FORMAT(' LAST 8 ELEMENS OF DIAG. STIFF. MATRIX = ')
      WRITE(5,203)STIFF
203   FORMAT(5D12.6)
      WRITE(5,204)
204   FORMAT(' PHI1 = ')
      WRITE(5,203)PHI1
      WRITE(5,205)
205   FORMAT(' PHI2 = ')
      WRITE(5,203)PHI2
      WRITE(5,206)
206   FORMAT(//' CURRENT RUN HAS F = 0 ')
      WRITE(5,207)
207   FORMAT(/' INPUT THE ROW AND COLUMN OF ALPHA'/)
      READ (4,208)I,J
208   FORMAT(2I2)
      WRITE(5,209)
209   FORMAT( /' INPUT THE ROW AND COLUMN OF BETA'/)
      READ (4,208) K,L
C
C..DETERMINE IF CASE I OR CASE II
C
      ICASE = 1
      IF((I.EQ.1 .AND. J.LE.3).AND.(K.EQ.2 .AND. L.GE.4)) ICASE = 2
      IF((I.EQ.1 .AND. J.GE.4).AND.(K.EQ.2. AND .L.LE.3)) ICASE = 2
      IF((I.EQ.2 .AND. J.LE.3).AND.(K.EQ.1. AND .L.GE.4)) ICASE = 2
      IF((I.EQ.2 .AND. J.GE.4).AND.(K.EQ.1. AND .L.LE.3)) ICASE = 2
1     WRITE(5,300)
300   FORMAT(/ ' ENTER VALUES OF SIG AND OMEGA  '/)
```

```
      READ(4,301) SIG,W
301   FORMAT(2D16.8)
      WRITE(5,302)
      READ(4,301)DSIG,DW
      WRITE(5,303)
302   FORMAT(/ ' ENTER VALUES OF DELTA SIG AND DELTA OMEGA  '/)
303   FORMAT(/ ' ENTER VALUES OF FINAL SIG AND FINAL OMEGA  '/)
      READ(4,301)FSIG,FW
 2    F(I,J) = 0.D0
      F(K,L) = 0.D0
      CALL BUILD(MASS,C,STIFF,F,PHI1,PHI2,SIG,W,WORK,U,X,Y)
      CALL COND(N,X,Y,MAXA)
      CALL CGAUSS(N,X,Y,IPVT)
      CALL DET(IPVT(N),N,X,Y,AR,AI)
      F(I,J) = 1.D0
      CALL BUILD(MASS,C,STIFF,F,PHI1,PHI2,SIG,W,WORK,U,X,Y)
      CALL COND(N,X,Y,MAXB)
      CALL CGAUSS(N,X,Y,IPVT)
      CALL DET(IPVT(N),N,X,Y,BR,BI)
      F(I,J) = 0.D0
      F(K,L) = 1.D0
      CALL BUILD(MASS,C,STIFF,F,PHI1,PHI2,SIG,W,WORK,U,X,Y)
      CALL COND(N,X,Y,MAXC)
      CALL CGAUSS(N,X,Y,IPVT)
      CALL DET(IPVT(N),N,X,Y,CR,CI)
      IF(ICASE.EQ.2)GO TO 20
      CALL CDIV(AR,AI,BR,BI,ALPHA1,ALPHA2)
      RATIO = (MAXB/MAXA)**N
      BR = ALPHA1*RATIO - 1.0D0
      BI = ALPHA2*RATIO
      CALL CDIV(AR,AI,CR,CI,ALPHA1,ALPHA2)
      RATIO = (MAXC/MAXA)**N
      CR = ALPHA1*RATIO - 1.0D0
      CI = ALPHA2*RATIO
      AR = 1.0D0
      AI = 0.0D0
      CALL SOLVE1(AR,AI,BR,BI,CR,CI,ALPHA,BETA)
C....START CHECK OUT MODE...CAN BE DELETED IF DESIRED.
      F(I,J) = ALPHA
      F(K,L) = BETA
      CALL BUILD(MASS,C,STIFF,F,PHI1,PHI2,SIG,W,WORK,U,X,Y)
       CALL COND(N,X,Y,MAXC)
      CALL CGAUSS(N,X,Y,IPVT)
      CALL DET(IPVT(N),N,X,Y,AR,AI)
      WRITE(4,30) SIG,W,ALPHA,BETA
30    FORMAT(' SIG,W,ALPHA,BETA = ',4D15.7)
      WRITE(4,31)
31    FORMAT(' X(1,1),Y(1,1),X(2,2),Y(2,2) ... = ',/)
      WRITE(4,32)(X(II,II),Y(II,II),  II = 1,N)
32    FORMAT(10D8.2)
C...END OF CHECK OUT MODE
      WRITE (8,400)SIG,W,ALPHA,BETA
      SIG = SIG + DSIG
      W = W + DW
      IF(SIG.LT.FSIG.AND.W.LT.FW)GO TO 2
```

```
      GO TO 3
C
C.....THIS IS CASE II
C
 20   F(I,J) = 1.0D0
      CALL BUILD(MASS,C,STIFF,F,PHI1,PHI2,SIG,W,WORK,U,X,Y)
      CALL CGAUSS(N,X,Y,IPVT)
      CALL DET(IPVT(N),N,X,Y,DR,DI)
      DR = DR - AR - BR - CR
      DI = DI - AI - BI - CI
      CALL SOLVE2(AR,AI,BR,BI,CR,CI,DR,DI,ALPHA1,BETA1,ALPHA2,BETA2)
      WRITE(8,500)SIG,W,ALPHA1,BETA1,ALPHA2,BETA2
      SIG = SIG + DSIG
      W = W + DW
      IF(SIG.LT.FSIG.AND.W.LT.FW)GO TO 2
  3   WRITE(5,600)
600   FORMAT( ' ENTER 1 TO CONTINUE ')
      READ(5,601) II
601   FORMAT(2I2)
      IF (II.EQ.1) GO TO 1
400   FORMAT(4D16.8)
500   FORMAT(6D16.8)
      STOP
      END
```

```
      SUBROUTINE BUILD(MASS,C,STIFF,F,PHI1,PHI2,SIG,W,WORK,U,X,Y)
      DOUBLE PRECISION MASS(4),C(6),STIFF(8),PHI1(10),PHI2(10)
      DOUBLE PRECISION F(2,6),X(10,10),Y(10,10),WORK(10,10),U(6,10)
      DOUBLE PRECISION SIG,W,C1,C2,C3,C4
      C1 = SIG/(SIG*SIG + W*W)
      C2 = SIG*SIG - W*W
      C3 = -W/(SIG*SIG + W*W)
      C4 = 2.D0*SIG*W
      DO 1 I = 1,10
      DO 1 J = 1,10
      X(I,J) = 0.0D0
      Y(I,J) = 0.0D0
 1    CONTINUE
       DO 2 I = 1,10
       DO 2 J = 1,6
       WORK(I,J) = PHI1(I)*F(1,J) + PHI2(I)*F(2,J)
 2     CONTINUE
       DO 3 J = 1,10
       U(1,J) = -SIG*PHI1(J)
       U(2,J) =      -PHI1(J)
       U(3,J) =  -C1*PHI1(J)
       U(4,J) = -SIG*PHI2(J)
       U(5,J) =      -PHI2(J)
       U(6,J) =  -C1*PHI2(J)
 3     CONTINUE
       CALL MMULTD(WORK,U,X,10,6,10)
       X(1,1) = MASS(1)*C2 + X(1,1)
       X(1,2) = MASS(2)*C2 + X(1,2)
       X(2,1) = MASS(3)*C2 + X(2,1)
       X(2,2) = MASS(4)*C2 + X(2,2)
       X(3,3) = C2 + STIFF(1) + X(3,3)
       X(4,4) = C2 + STIFF(2) + X(4,4)
       DO 5 I = 5,10
       KK = I-2
       KKK = I-4
       X(I,I) = C2 + STIFF(KK) + X(I,I) + SIG*C(KKK)
 5     CONTINUE
       DO 6 J = 1,10
       U(1,J) = -W*PHI1(J)
       U(2,J) =  0.0D0
       U(3,J) = -C3*PHI1(J)
       U(4,J) = -W*PHI2(J)
       U(5,J) =  0.0D0
       U(6,J) = -C3*PHI2(J)
 6     CONTINUE
       CALL MMULTD(WORK,U,Y,10,6,10)
       Y(1,1) = MASS(1)*C4 + Y(1,1)
       Y(1,2) = MASS(2)*C4 + Y(1,2)
       Y(2,1) = MASS(3)*C4 + Y(2,1)
       Y(2,2) = MASS(4)*C4 + Y(2,2)
       Y(3,3) = C4  + Y(3,3)
       Y(4,4) = C4  + Y(4,4)
       DO 8 I = 5,10
       KK = I-4
       Y(I,I) = C4 + C(KK)*W + Y(I,I)
```

```
  8       CONTINUE
          RETURN
          END

      SUBROUTINE MMULTD(A,B,C,L,M,N)
      DOUBLE PRECISION A(L,M),B(M,N),C(L,N)
      DO 1 I = 1,L
      DO 1 J = 1,N
      C(I,J) = 0.0D0
      DO 1 K = 1,M
      C(I,J) = C(I,J) + A(I,K)*B(K,J)
    1 CONTINUE
      RETURN
      END

      SUBROUTINE CDIV(X,Y,U,V,AR,AI)
      DOUBLE PRECISION X,Y,U,V,AR,AI,D
      D = X*X + Y*Y
      IF (D.EQ.0.0D0) RETURN
      AR = (X*U + Y*V)/D
      AI = (X*V - Y*U)/D
      RETURN
      END

      SUBROUTINE COND(N,X,Y,MAX)
      DOUBLE PRECISION X(N,N),Y(N,N),MAX
C.....IF TROUBLE EXISTS WITH CGAUSS WE MAY HAVE TO SCALE
C.... THE ELEMENTS OF THE MATRIX BEFORE DOING A GAUSSION
C.... ELEMINATION.  OTHERWISE, THIS IS A DUMMY ROUTINE AND
C.... MAY BE IGNORED.  CURRENT VERSION SCALES EACH ELEMENT BY MAX.
      MAX = 100.0D0
      DO 1 I = 1,N
      DO 1 J = 1,N
      X(I,J) = X(I,J)/MAX
      Y(I,J) = Y(I,J)/MAX
1     CONTINUE
      RETURN
      END

      SUBROUTINE CGAUSS(N,X,Y,IPVT)
      DIMENSION IPVT(N)
      DOUBLE PRECISION X(N,N),Y(N,N)
      DOUBLE PRECISION D1,D2,T1,T2,T
      IF(N.EQ.1) RETURN
      IPVT(N) = 1
      NM1 = N-1
      DO 35 K = 1,NM1
      KP1 = K+1
      M = K
      DO 15 I = KP1,N
      D1 = X(I,K)*X(I,K) + Y(I,K)*Y(I,K)
      D2 = X(M,K)*X(M,K) + Y(M,K)*Y(M,K)
```

```
      IF(DABS(D1).GT.DABS(D2)) M = I
 15   CONTINUE
      IPVT(K) = M
      IF(M.NE.K) IPVT(N) = -IPVT(N)
      T1 = X(M,K)
      T2 = Y(M,K)
      X(M,K) = X(K,K)
      Y(M,K) = Y(K,K)
      X(K,K) = T1
      Y(K,K) = T2
      T = T1*T1 + T2*T2
      IF(T.EQ.0.0D0)GO TO 35
      DO 20 I = KP1,N
      D1 = -(X(I,K)*T1 + Y(I,K)*T2)/T
      Y(I,K) = (X(I,K)*T2 - Y(I,K)*T1)/T
      X(I,K) = D1
 20   CONTINUE
      DO 30 J = KP1,N
      T1 = X(M,J)
      T2 = Y(M,J)
      X(M,J) = X(K,J)
      Y(M,J) = Y(K,J)
      X(K,J) = T1
      Y(K,J) = T2
      T = T1*T1 + T2*T2
      IF(T.EQ.0.0D0) GO TO 30
      DO 25 I = KP1,N
      D1 = X(I,K)*T1 - Y(I,K)*T2
      D2 = X(I,K)*T2 + Y(I,K)*T1
      X(I,J) = X(I,J) + D1
      Y(I,J) = Y(I,J) + D2
 25   CONTINUE
 30   CONTINUE
 35   CONTINUE
      RETURN
      END

      SUBROUTINE SOLVE1(AR,AI,BR,BI,CR,CI,X1,X2)
      DOUBLE PRECISION AR,AI,BR,BI,CR,CI,X1,X2,DET
      DET = BR*CI - BI*CR
      IF(DABS(DET).LT.1.0D-37)GO TO 10
      X1 = -CI/DET
      X2 =  BI/DET
      RETURN
 10   WRITE(5,100)
100   FORMAT(' TROUBLE, LET ALPHA = BETA = 0 ')
      X1 = 0.0D0
      X2 = 0.0D0
      RETURN
      END
```

```
      SUBROUTINE SOLVE2(AR,AI,BR,BI,CR,CI,DR,DI,X1,X2,X3,X4)
      DOUBLE PRECISION AR,AI,BR,BI,CR,CI,DR,DI,X1,X2,X3,X4
      DOUBLE PRECISION A(8),B(3),D4,BB,CC,DISC,DET,BIGA,BIGB
      A(8) = DR
      A(7) = DI
      A(6) = CR
      A(5) = CI
      A(4) = BR
      A(3) = BI
      A(2) = AR
      A(1) = AI
C
C  A(8)XY + A(6)x + A(4)Y + A(2) = 0
C  A(7)XY + A(5)X + A(3)Y + A(1) = 0
C
222   CONTINUE
      D4 = DABS(A(7)) + DABS(A(8))
      IF (D4.EQ.0)GO TO 15
C...GET COEFFICIENTS OF LINEAR EQN BY ELIMINATING NONLINEAR TERM
      B(1) = A(6)*A(7) - A(5)*A(8)
      B(2) = A(4)*A(7) - A(3)*A(8)
      B(3) = A(2)*A(7) - A(1)*A(8)
      IF(B(1).EQ.0.0 .AND. B(2).EQ.0.0)GO TO 20
      M = 1
      BIGB = DABS(B(1))
      IF(BIGB.LT.DABS(B(2))M = 2
      D4 = B(M)
      DO 1 I = 1,3
1     B(I) = B(I)/D4
      L = 7
      BIGA = DABS(A(7))
      IF(BIGA.LE.DABS(A(8))L = 8
      D4 = A(L)
      DO 2 I = 1,8
2     A(I) = A(I)/D4
      IF(L.EQ.7) A(2) = A(1)
      IF(L.EQ.7) A(4) = A(3)
      IF(L.EQ.7) A(6) = A(5)
      IF(M.EQ.2) GOTO 5
      IF (B(2).EQ.0.0)GOTO 10
      BB = (B(3) + B(2)*A(6) - A(4))/B(2)
      CC = (B(3)*A(6) - A(2))/B(2)
      DISC = BB*BB - 4.*CC
      IF(DISC.LT.0.0) RETURN
      X2 = (-BB + DSQRT(DISC))/2.
      X4 = -X2 - BB
      X1 = -B(2)*X2 - B(3)
      X3 = -B(2)*X4 - B(3)
      RETURN
10    DIV = A(4) - B(3)
      IF(DIV.EQ.0)GO TO 20
C...THERE IS ONLY ONE ROOT
      X1 = - B(3)
      X2 = (A(6)*B(3) - A(2))/DIV
```

```
      RETURN
5     IF(B(1).EQ.0.0)GO TO 11
      BB = (B(3) + B(1)*A(4) - A(6))/B(1)
      CC = (B(3)*A(4) - A(2))/B(1)
      DISC = BB*BB - 4.*CC
      IF(DISC.LT.0.0)RETURN
      X1 = (-BB + DSQRT(DISC))/2.
      X3 = -X1 - BB
      X2 = -B(1)*X1 - B(3)
      X4 = -B(1)*X3 - B(3)
      RETURN
11    DIV = B(3) - A(6)
      IF(DIV.EQ.0)GOTO 20
C....ONLY ONE ROOT
      X2 = -B(3)
      X1 = (A(2) - A(4)*B(3))/DIV
      RETURN
15    CONTINUE
      GO TO 111
20    CONTINUE
      RETURN
      END
```

```
M11,M12,M21,M22 =
.347700D+05-.730800D+05-.730800D+05 .850600D+07
LAST 6 ELEMENTS OF DIAG. DAMPING MATRIX =
.141400D-01 .182900D-01 .284600D-01 .383100D-01 .879800D-01 .961300D-01
LAST 8 ELEMENS OF DIAG. STIFF. MATRIX =
.219000D-01 .829000D-01 .200000D+01 .334500D+01 .810000D+01
.146800D+02 .774100D+02 .924100D+02
PHI1 =
.100000D+01 .100000D+01-.912000D-03-.130000D-03-.140000D-03
.407000D-03 .285000D-02 .713000D-02-.800000D-07-.172000D-04
PHI2 =
.100000D+01 .100000D+01 .344000D-02 .548000D-04 .360000D-02
.347000D-02 .395000D-02-.246000D-02-.115000D-08-.413000D-06


CURRENT RUN HAS F = 0

INPUT THE ROW AND COLUMN OF ALPHA
1,1


INPUT THE ROW AND COLUMN OF BETA
1,2


ENTER VALUES OF SIG AND OMEGA
0.,.1


ENTER VALUES OF DELTA SIG AND DELTA OMEGA
0.,.1


ENTER VALUES OF FINAL SIG AND FINAL OMEGA
1.,.25

SIG,W,ALPHA,BETA =    .0000000D+01   .1000000D+00   .1534355D-02  -.3428732D+03
X(1,1),Y(1,1),X(2,2),Y(2,2) ... =

.11D+02-.15D-05 .69D+01-.12D-03 .88D-03 .15D-07 .73D-03 .32D-26 .20D-01 .14D-04
.33D-01 .18D-04 .81D-01 .28D-04 .35D-03 .92D-07 .77D+00 .88D-04-.62D-09 .14D-12
SIG,W,ALPHA,BETA =    .0000000D+01   .2000000D+00   .2109157D-01  -.1267784D+04
X(1,1),Y(1,1),X(2,2),Y(2,2) ... =

.42D+02-.42D-04-.58D+02-.36D-02-.16D-02 .10D-06 .43D-03 .00D+01 .20D-01 .28D-04
.33D-01 .37D-04 .81D-01 .57D-04 .35D-03 .18D-06 .77D+00 .18D-03 .74D-09-.15D-12
ENTER 1 TO CONTINUE
 STOP

[A]                    DARPA   SAMPLE RUN
```

```
      DOUBLE PRECISION MASS(4),C(6),STIFF(8),PHI1(10),CC(10)
      DOUBLE PRECISION SIG,W,DSIG,DW,FSIG,FW,AR,AI,BR,BI,CR,CI,DR,DI
      DOUBLE PRECISION ALPHA,BETA,DELR,DELI,DELT,S2R,S2I,T,SUMR,SUMI
      READ(6,100)C
      READ(6,100)STIFF
      READ(6,100)PHI1
      READ(6,100)CC
      READ(6,100)MASS
 100  FORMAT(10D11.4)
      WRITE(5,200)MASS
 200  FORMAT(' M11,M12,M21,M22 = ',/,4D12.6)
      CC(1) = 0.0D0
      CC(2) = 0.0D0
      DO 16 I = 3,8
      K = I-2
      CC(I) = C(K)
 16   CONTINUE
      WRITE(5,201)
 201  FORMAT(' LAST 8 ELEMENTS OF DIAG. DAMPING MATRIX = ')
      WRITE(5,202)(CC(I),I=1,8)
 202  FORMAT(5D15.7)
      WRITE(5,203)
 203  FORMAT(' LAST 8 ELEMENTS OF DIAG. STIFF. MATRIX = ')
      WRITE(5,202)STIFF
      WRITE(5,204)
 204  FORMAT(' PHI1 = ')
      WRITE(5,202)PHI1
 1    WRITE(5,300)
 300  FORMAT(// ' ENTER VALUES OF SIG AND OMEGA  '/)
      READ(4,301) SIG,W
 301  FORMAT(2D16.8)
      WRITE(5,302)
      READ(4,301)DSIG,DW
      WRITE(5,303)
 302  FORMAT(// ' ENTER VALUES OF DELTA SIG AND DELTA OMEGA  '/)
 303  FORMAT(// ' ENTER VALUES OF FINAL SIG AND FINAL OMEGA  '/)
      READ(4,301)FSIG,FW
 2    AR = 1.0D0
      AI = 0.0D0
      DELT =MASS(1)*MASS(4) - MASS(2)*MASS(3)
C
C...TRYING TO PREVENT OVERFLOW
C
      DELR = 1.0D0
      DELI = 0.0D0
      CALL CMULT(SIG,W,SIG,W,S2R,S2I)
      CALL CDIV(S2R,S2I,1.0D0,0.0D0,AR,AI)
      T = (MASS(1) - 2.D0*MASS(2) + MASS(4))/DELT
      SUMR = T*AR
      SUMI = T*AI
      DO 5 I = 1,8
      CALL QUAD(CC(I),STIFF(I),BR,BI,SIG,W)
      CALL CDIV(BR,BI,1.0D0,0.0D0,CR,CI)
      SUMR = PHI1(I+2)*PHI1(I+2)*CR + SUMR
      SUMI = PHI1(I+2)*PHI1(I+2)*CI + SUMI
```

```
  5     CONTINUE
        CR = -SUMR
        CI = -SUMI
        CALL CMULT(CR,CI,SIG,W,BR,BI)
        AR = DELR
        AI = DELI
        CALL SOLVE1(AR,AI,BR,BI,CR,CI,ALPHA,BETA)
        DELR = BR*CI - BI*CR
        WRITE(4,211)SIG,W,DELR,ALPHA,BETA
        WRITE(8,212)SIG,W,ALPHA,BETA
 211    FORMAT(' SIG,W,DET,ALPHA,BETA = ',2D8.2,3D12.5)
 212    FORMAT(5D16.8)
        SIG = SIG + DSIG
        W   = W + DW
        IF ( (SIG.LE.FSIG).AND.(W.LE.FW) ) GO TO 2
        WRITE(4,312)
 312    FORMAT(' ENTER 1 TO CONTINUE'/)
        READ(5,213)II
 213    FORMAT(I2)
        IF(II.EQ.1)GO TO 1
        STOP
        END
C
C
        SUBROUTINE CMULT(X,Y,U,V,AR,AI)
        DOUBLE PRECISION X,Y,U,V,AR,AI
        AR = X*U - Y*V
        AI = X*V + Y*U
        RETURN
        END
C
C
        SUBROUTINE CDIV(X,Y,U,V,AR,AI)
        DOUBLE PRECISION X,Y,U,V,AR,AI,D
        D = X*X + Y*Y
        IF (D.EQ.0.0D0) RETURN
        AR = (X*U + Y*V)/D
        AI = (X*V - Y*U)/D
        RETURN
        END
C
C
        SUBROUTINE QUAD(A,B,AR,AI,SIG,W)
        DOUBLE PRECISION A,B,AR,AI,SIG,W
        AR = SIG*SIG - W*W + A*SIG + B
        AI = 2.0D0*W*SIG + A*W
        RETURN
        END
C
C
      SUBROUTINE SOLVE1(AR,AI,BR,BI,CR,CI,X1,X2)
      DOUBLE PRECISION AR,AI,BR,BI,CR,CI,X1,X2,DET
      DET = BR*CI - BI*CR
      IF(DABS(DET).LT.1.0D-37)GO TO 10
      X1 = -CI/DET
```

```
      X2 =  BI/DET
      RETURN
10    WRITE(5,100)
100   FORMAT(' TROUBLE, LET ALPHA = BETA = 0 ')
      X1 = 0.0D0
      X2 = 0.0D0
      RETURN
      END
```

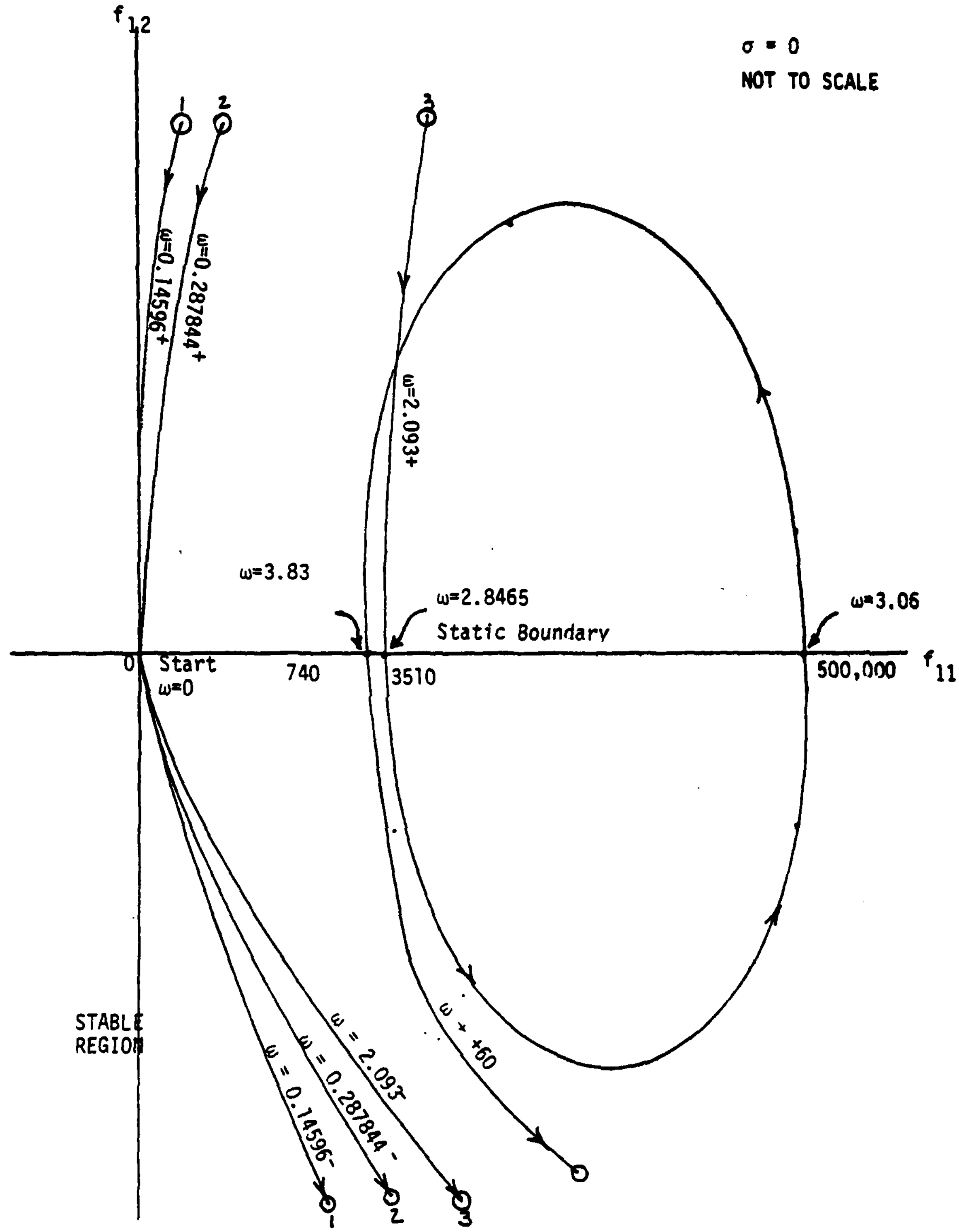


Fig 1. PARAMETER PLANE - f_{11} vs. f_{12}

```
LDARPA

M11,M12,M21,M22 =
.347700D+05-.730800D+05-.730800D+05 .850600D+07
LAST 8 ELEMENTS OF DIAG. DAMPING MATRIX =
  .0000000D+01   .0000000D+01   .1414000D-01   .1829000D-01   .2846000D-01
  .3831000D-01   .8798000D-01   .9613000D-01
LAST 8 ELEMENTS OF DIAG. STIFF. MATRIX =
  .2190000D-01   .8290000D-01   .2000000D+01   .3345000D+01   .8100000D+01
  .1468000D+02   .7741000D+02   .9241000D+02
PHI1 =
  .1000000D+01   .1000000D+01  -.9120000D-03  -.1300000D-03  -.1400000D-03
 -.4070000D-03   .2850000D-02   .7180000D-02  -.8000000D-07  -.1720000D-04


ENTER VALUES OF SIG AND OMEGA
0.,.1


ENTER VALUES OF DELTA SIG AND DELTA OMEGA
0.,.1


ENTER VALUES OF FINAL SIG AND FINAL OMEGA
1.,1.01

SIG,W,DET,ALPHA,BETA =  .00D+01 .10D+00 -.85061D-06  .15344D-02 -.34287D+03
SIG,W,DET,ALPHA,BETA =  .00D+01 .20D+00 -.12443D-06  .21092D-01 -.12678D+04
SIG,W,DET,ALPHA,BETA =  .00D+01 .30D+00 -.35160D-07  .11299D+00 -.29210D+04
SIG,W,DET,ALPHA,BETA =  .00D+01 .40D+00 -.14222D-07  .37731D+00 -.53034D+04
SIG,W,DET,ALPHA,BETA =  .00D+01 .50D+00 -.70478D-08  .96788D+00 -.84228D+04
SIG,W,DET,ALPHA,BETA =  .00D+01 .60D+00 -.39263D-08  .21292D+01 -.12362D+05
SIG,W,DET,ALPHA,BETA =  .00D+01 .70D+00 -.23623D-08  .42353D+01 -.17214D+05
SIG,W,DET,ALPHA,BETA =  .00D+01 .80D+00 -.14984D-08  .78681D+01 -.23107D+05
SIG,W,DET,ALPHA,BETA =  .00D+01 .90D+00 -.98573D-09  .13954D+02 -.30216D+05
SIG,W,DET,ALPHA,BETA =  .00D+01 .10D+01 -.66453D-09  .24027D+02 -.38792D+05
ENTER 1 TO CONTINUE

 STOP
```

LDARPA SAMPLE RUN

SIGMA	OMEGA	F(1,1)	F(1,2)
.00000000D+01	.10000000D-01	.14564866D-06	-.33435819D+01
.00000000D+01	.20000000D-01	.23325464D-05	-.13380185D+02
.00000000D+01	.30000000D-01	.11827427D-04	-.30123155D+02
.00000000D+01	.40000000D-01	.37468619D-04	-.53620865D+02
.00000000D+01	.50000000D-01	.91773331D-04	-.83911786D+02
.00000000D+01	.60000000D-01	.19112995D-03	-.12108379D+03
.00000000D+01	.70000000D-01	.35614535D-03	-.16526630D+03
.00000000D+01	.80000000D-01	.61230523D-03	-.21666854D+03
.00000000D+01	.90000000D-01	.99133676D-03	-.27564861D+03
.00000000D+01	.10000000D+00	.15343547D-02	-.34287322D+03
.00000000D+01	.11000000D+00	.23003410D-02	-.41974390D+03
.00000000D+01	.12000000D+00	.33948322D-02	-.50980860D+03
.00000000D+01	.13000000D+00	.51124052D-02	-.62547862D+03
.00000000D+01	.14000000D+00	.97218399D-02	-.86231907D+03
.00000000D+01	.14500000D+00	.59891211D-01	-.21400273D+04
.00000000D+01	.14510000D+00	.69839019D-01	-.23109241D+04
.00000000D+01	.14520000D+00	.83464236D-01	-.25263044D+04
.00000000D+01	.14530000D+00	.10298832D+00	-.28062626D+04
.00000000D+01	.14540000D+00	.13267561D+00	-.31851415D+04
.00000000D+01	.14550000D+00	.18164666D+00	-.37268830D+04
.00000000D+01	.14560000D+00	.27261384D+00	-.45656735D+04
.00000000D+01	.14570000D+00	.47693608D+00	-.60389280D+04
.00000000D+01	.14580000D+00	.11326957D+01	-.93064754D+04
.00000000D+01	.14590000D+00	.67774853D+01	-.22764691D+05
.00000000D+01	.14600000D+00	.19655781D+02	.38767823D+05
.00000000D+01	.14610000D+00	.12399394D+01	.97370053D+04
.00000000D+01	.14620000D+00	.37003492D+00	.53191885D+04
.00000000D+01	.14630000D+00	.16308971D+00	.35313125D+04
.00000000D+01	.14640000D+00	.85934104D-01	.25633299D+04
.00000000D+01	.14650000D+00	.50050446D-01	.19562509D+04
.00000000D+01	.14660000D+00	.31009480D-01	.15398083D+04
.00000000D+01	.14670000D+00	.19989451D-01	.12362853D+04
.00000000D+01	.14680000D+00	.13213922D-01	.10051549D+04
.00000000D+01	.14690000D+00	.88632056D-02	.82321138D+03
.00000000D+01	.14700000D+00	.59804427D-02	.67620981D+03
.00000000D+01	.14710000D+00	.40275020D-02	.55492147D+03
.00000000D+01	.14720000D+00	.26851706D-02	.45310456D+03
.00000000D+01	.14730000D+00	.17557217D-02	.36638625D+03
.00000000D+01	.14740000D+00	.11122183D-02	.29161187D+03
.00000000D+01	.14750000D+00	.67068735D-03	.22644830D+03
.00000000D+01	.14760000D+00	.37414568D-03	.16913295D+03
.00000000D+01	.14770000D+00	.18307318D-03	.11830935D+03
.00000000D+01	.14780000D+00	.69541088D-04	.72916577D+02
.00000000D+01	.14790000D+00	.13488117D-04	.32112937D+02
.00000000D+01	.14800000D+00	.29858543D-06	-.47779050D+01
.00000000D+01	.15000000D+00	.17800848D-02	-.36889275D+03
.00000000D+01	.16000000D+00	.67900850D-02	-.72027024D+03
.00000000D+01	.17000000D+00	.99206569D-02	-.87035862D+03
.00000000D+01	.18000000D+00	.13169896D-01	-.10024931D+04
.00000000D+01	.19000000D+00	.16848296D-01	-.11335024D+04
.00000000D+01	.20000000D+00	.21091574D-01	-.12677838D+04
.00000000D+01	.21000000D+00	.25998017D-01	-.14070161D+04
.00000000D+01	.22000000D+00	.31656863D-01	-.15520051D+04
.00000000D+01	.23000000D+00	.38157347D-01	-.17032170D+04
.00000000D+01	.24000000D+00	.45593350D-01	-.18609998D+04

SIGMA	OMEGA	F(1,1)	F(1,2)
.00000000D+01	.25000000D+00	.54068640D-01	-.20256965D+04
.00000000D+01	.26000000D+00	.63709616D-01	-.21978706D+04
.00000000D+01	.27000000D+00	.74718126D-01	-.23790440D+04
.00000000D+01	.28000000D+00	.87826588D-01	-.25780078D+04
.00000000D+01	.28700000D+00	.11407950D+00	-.29370909D+04
.00000000D+01	.28710000D+00	.11689545D+00	-.29731040D+04
.00000000D+01	.28720000D+00	.12058962D+00	-.30197012D+04
.00000000D+01	.28730000D+00	.12567636D+00	-.30827161D+04
.00000000D+01	.28740000D+00	.13316658D+00	-.31732338D+04
.00000000D+01	.28750000D+00	.14535172D+00	-.33152192D+04
.00000000D+01	.28760000D+00	.16873042D+00	-.35718776D+04
.00000000D+01	.28770000D+00	.23125858D+00	-.41816388D+04
.00000000D+01	.28780000D+00	.75119912D+00	-.75365544D+04
.00000000D+01	.28790000D+00	.17327799D-01	.11446289D+04
.00000000D+01	.28800000D+00	.23279390D-01	-.13267116D+04
.00000000D+01	.28810000D+00	.46085180D-01	-.18666766D+04
.00000000D+01	.28820000D+00	.58556285D-01	-.21041316D+04
.00000000D+01	.28830000D+00	.66257989D-01	-.22382211D+04
.00000000D+01	.28840000D+00	.71480591D-01	-.23247466D+04
.00000000D+01	.30000000D+00	.11299333D+00	-.29210337D+04
.00000000D+01	.50000000D+00	.96788292D+00	-.84228344D+04
.00000000D+01	.70000000D+00	.42353357D+01	-.17213891D+05
.00000000D+01	.90000000D+00	.13953811D+02	-.30216375D+05
.00000000D+01	.11000000D+01	.40809051D+02	-.49194179D+05
.00000000D+01	.13000000D+01	.12986081D+03	-.78063751D+05
.00000000D+01	.15000000D+01	.40671942D+03	-.12396795D+06
.00000000D+01	.17000000D+01	.19663688D+04	-.228541650D+06
.00000000D+01	.19000000D+01	.97089372D+04	-.37169729D+06
.00000000D+01	.20900000D+01	.48834018D+07	-.48974132D+07
.00000000D+01	.20910000D+01	.54074079D+07	-.37352732D+07
.00000000D+01	.20920000D+01	.57967006D+07	-.21985565D+07
.00000000D+01	.20930000D+01	.59705091D+07	-.40837459D+06
.00000000D+01	.20940000D+01	.58879446D+07	.14240560D+07
.00000000D+01	.20950000D+01	.55700729D+07	.30718458D+07
.00000000D+01	.20960000D+01	.50880317D+07	.43781987D+07
.00000000D+01	.20970000D+01	.45282139D+07	.52927990D+07
.00000000D+01	.20980000D+01	.39620073D+07	.58501557D+07
.00000000D+01	.20990000D+01	.34341702D+07	.61252795D+07
.00000000D+01	.22000000D+01	.27463862D+05	.79201906D+06
.00000000D+01	.22500000D+01	.13774664D+05	.53790155D+06
.00000000D+01	.23000000D+01	.85686149D+04	.40357793D+06
.00000000D+01	.23500000D+01	.60248148D+04	.31928990D+06
.00000000D+01	.24000000D+01	.45908148D+04	.26048193D+06
.00000000D+01	.24500000D+01	.37084945D+04	.21625722D+06
.00000000D+01	.25000000D+01	.31377357D+04	.18099226D+06
.00000000D+01	.25500000D+01	.27632936D+04	.15143085D+06
.00000000D+01	.26000000D+01	.25276414D+04	.12547728D+06
.00000000D+01	.26500000D+01	.24052460D+04	.10161035D+06
.00000000D+01	.27000000D+01	.23941849D+04	.78536382D+05
.00000000D+01	.27500000D+01	.25192944D+04	.54902433D+05
.00000000D+01	.28000000D+01	.28519092D+04	.28919939D+05
.00000000D+01	.28500000D+01	.35730315D+04	-.23796301D+04
.00000000D+01	.29000000D+01	.51956836D+04	-.45114368D+05
.00000000D+01	.29500000D+01	.96042674D+04	-.11541321D+06
.00000000D+01	.30000000D+01	.28432593D+05	-.27448942D+06

SIGMA	OMEGA	F(1,1)	F(1,2)
.00000000D+01	.30500000D+01	.32181669D+06	-.67218260D+06
.00000000D+01	.31000000D+01	.64136293D+05	.56653739D+06
.00000000D+01	.31500000D+01	.14457474D+05	.30030866D+06
.00000000D+01	.32000000D+01	.65158104D+04	.20698349D+06
.00000000D+01	.32500000D+01	.38962479D+04	.15828978D+06
.00000000D+01	.33000000D+01	.27114103D+04	.12708258D+06
.00000000D+01	.33500000D+01	.20703151D+04	.10450872D+06
.00000000D+01	.34000000D+01	.16805598D+04	.86842934D+05
.00000000D+01	.34500000D+01	.14233180D+04	.72244497D+05
.00000000D+01	.35000000D+01	.12428425D+04	.59697176D+05
.00000000D+01	.35500000D+01	.11101155D+04	.48592269D+05
.00000000D+01	.36000000D+01	.10087899D+04	.38541625D+05
.00000000D+01	.36500000D+01	.92906418D+03	.29285134D+05
.00000000D+01	.37000000D+01	.86475665D+03	.20641308D+05
.00000000D+01	.37500000D+01	.81180146D+03	.12479207D+05
.00000000D+01	.38000000D+01	.76742820D+03	.47016666D+04
.00000000D+01	.38500000D+01	.72969116D+03	-.27651680D+04
.00000000D+01	.39000000D+01	.69718751D+03	-.99786051D+04
.00000000D+01	.39500000D+01	.66888242D+03	-.16983919D+05
.00000000D+01	.40000000D+01	.64399690D+03	-.23817444D+05
.00000000D+01	.40500000D+01	.62193398D+03	-.30508745D+05
.00000000D+01	.41000000D+01	.60222876D+03	-.37082180D+05
.00000000D+01	.41500000D+01	.58451395D+03	-.43558031D+05
.00000000D+01	.42000000D+01	.56849558D+03	-.49953361D+05
.00000000D+01	.42500000D+01	.55393556D+03	-.56282646D+05
.00000000D+01	.43000000D+01	.54063901D+03	-.62558267D+05
.00000000D+01	.43500000D+01	.52844484D+03	-.68790892D+05
.00000000D+01	.44000000D+01	.51721870D+03	-.74989770D+05
.00000000D+01	.44500000D+01	.50684759D+03	-.81162973D+05
.00000000D+01	.45000000D+01	.49723579D+03	-.87317581D+05
.00000000D+01	.45500000D+01	.48830163D+03	-.93459835D+05
.00000000D+01	.46000000D+01	.47997496D+03	-.99595266D+05
.00000000D+01	.46500000D+01	.47219519D+03	-.10572879D+06
.00000000D+01	.47000000D+01	.46490969D+03	-.11186481D+06
.00000000D+01	.47500000D+01	.45807250D+03	-.11800726D+06
.00000000D+01	.48000000D+01	.45164330D+03	-.12415969D+06
.00000000D+01	.48500000D+01	.44558655D+03	-.13032529D+06
.00000000D+01	.49000000D+01	.43987080D+03	-.13650697D+06
.00000000D+01	.49500000D+01	.43446811D+03	-.14270735D+06

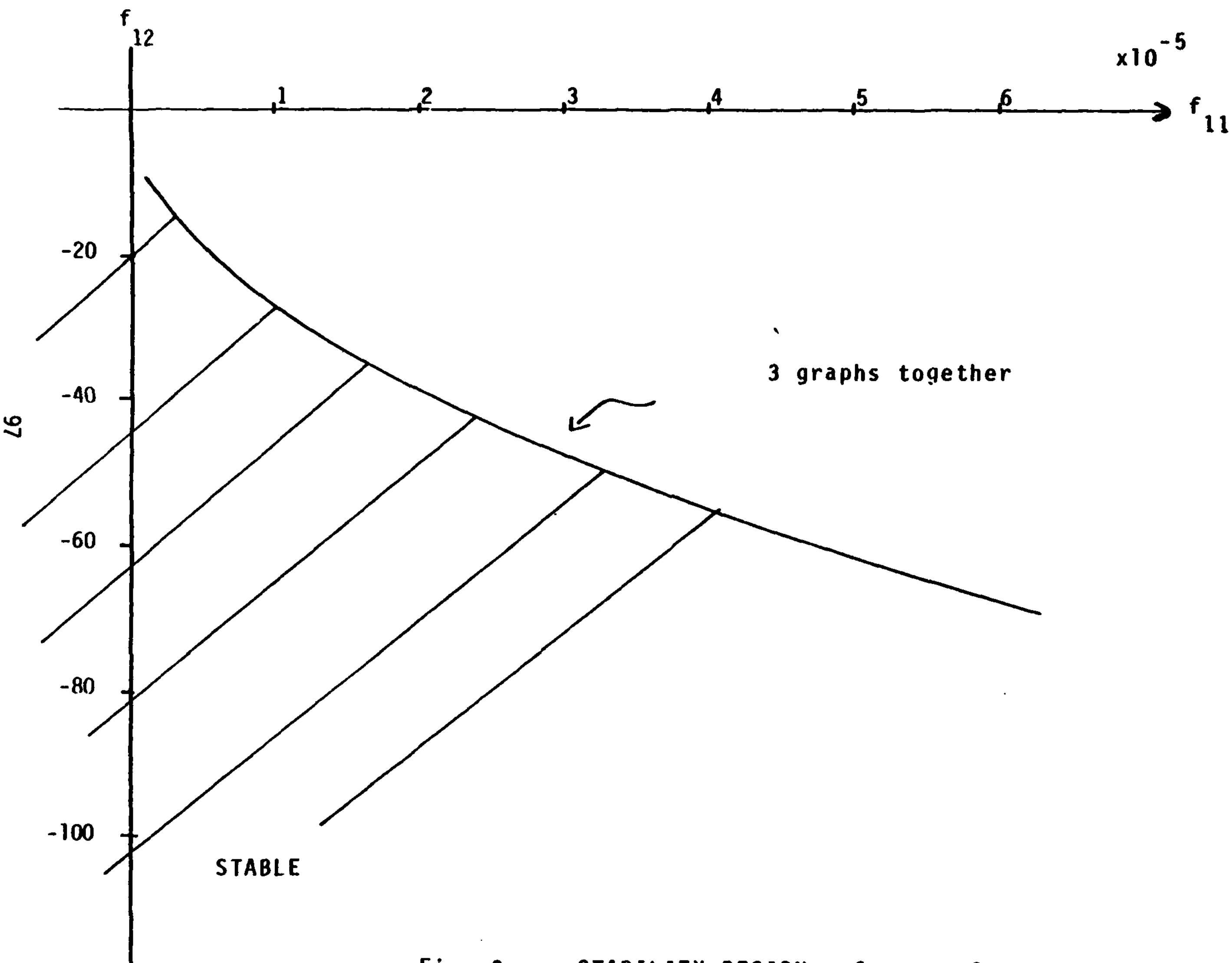


Fig. 2. STABILITY REGION - f_{11} vs. f_{12}

SIGMA	OMEGA	F(1,1)	F(1,2)
.00000000D+01	.10000000D-01	.14564866D-06	-.33435819D+01
.00000000D+01	.11000000D-01	.21325792D-06	-.40458567D+01
.00000000D+01	.12000000D-01	.30205783D-06	-.48150639D+01
.00000000D+01	.13000000D-01	.41607529D-06	-.56512170D+01
.00000000D+01	.14000000D-01	.55968904D-06	-.65543310D+01
.00000000D+01	.15000000D-01	.73763007D-06	-.75244219D+01
.00000000D+01	.16000000D-01	.95498204D-06	-.85615071D+01
.00000000D+01	.17000000D-01	.12171818D-05	-.96656052D+01
.00000000D+01	.18000000D-01	.15300200D-05	-.10836736D+02
.00000000D+01	.19000000D-01	.18996416D-05	-.12074922D+02
.00000000D+01	.20000000D-01	.23325464D-05	-.13380185D+02
.00000000D+01	.21000000D-01	.28355899D-05	-.14752550D+02
.00000000D+01	.22000000D-01	.34159840D-05	-.16192041D+02
.00000000D+01	.23000000D-01	.40812976D-05	-.17698687D+02
.00000000D+01	.24000000D-01	.48394577D-05	-.19272517D+02
.00000000D+01	.25000000D-01	.56987498D-05	-.20913560D+02
.00000000D+01	.26000000D-01	.66678194D-05	-.22621848D+02
.00000000D+01	.27000000D-01	.77556726D-05	-.24397416D+02
.00000000D+01	.28000000D-01	.89716771D-05	-.26240298D+02
.00000000D+01	.29000000D-01	.10325564D-04	-.28150531D+02
.00000000D+01	.30000000D-01	.11827427D-04	-.30128155D+02
.00000000D+01	.31000000D-01	.13487727D-04	-.32173210D+02
.00000000D+01	.32000000D-01	.15317291D-04	-.34285739D+02
.00000000D+01	.33000000D-01	.17327312D-04	-.36465787D+02
.00000000D+01	.34000000D-01	.19529356D-04	-.38713401D+02
.00000000D+01	.35000000D-01	.21935356D-04	-.41028629D+02
.00000000D+01	.36000000D-01	.24557620D-04	-.43411523D+02
.00000000D+01	.37000000D-01	.27408831D-04	-.45862136D+02
.00000000D+01	.38000000D-01	.30502047D-04	-.48380525D+02
.00000000D+01	.39000000D-01	.33850704D-04	-.50966748D+02
.00000000D+01	.40000000D-01	.37468619D-04	-.53620865D+02
.00000000D+01	.41000000D-01	.41369991D-04	-.56342941D+02
.00000000D+01	.42000000D-01	.45569404D-04	-.59133042D+02
.00000000D+01	.43000000D-01	.50081829D-04	-.61991239D+02
.00000000D+01	.44000000D-01	.54922628D-04	-.64917602D+02
.00000000D+01	.45000000D-01	.60107555D-04	-.67912209D+02
.00000000D+01	.14802000D+00	.18017429D-05	-.11736791D+02
.00000000D+01	.14804000D+00	.45081725D-05	-.18565344D+02
.00000000D+01	.14806000D+00	.83505005D-05	-.25267295D+02
.00000000D+01	.14808000D+00	.13265135D-04	-.31846236D+02
.00000000D+01	.14810000D+00	.19192032D-04	-.38305622D+02
.00000000D+01	.14812000D+00	.26074478D-04	-.44648782D+02
.00000000D+01	.14814000D+00	.33858884D-04	-.50878921D+02
.00000000D+01	.14816000D+00	.42494600D-04	-.56999128D+02
.00000000D+01	.14818000D+00	.51933736D-04	-.63012381D+02
.00000000D+01	.14820000D+00	.62130997D-04	-.68921550D+02
.00000000D+01	.28792400D+00	.24204391D-05	-.13528185D+02
.00000000D+01	.28792420D+00	.54276346D-05	-.20258056D+02
.00000000D+01	.28792440D+00	.96088771D-05	-.26954344D+02
.00000000D+01	.28792460D+00	.14946545D-04	-.33617303D+02
.00000000D+01	.28792480D+00	.21423281D-04	-.40247178D+02
.00000000D+01	.28792500D+00	.29021984D-04	-.46844218D+02
.00000000D+01	.28792520D+00	.37725807D-04	-.53408664D+02
.00000000D+01	.28792540D+00	.47518159D-04	-.59940759D+02
.00000000D+01	.28792560D+00	.58382686D-04	-.66440740D+02

ENLARGEMENT OF STABLE REGION (SIGMA = 0)   PAGE 1

SIGMA	OMEGA	F(1,1)	F(1,2)
.00000000D+01	.28792580D+00	.70303280D-04	-.72908845D+02
.00000000D+01	.28792600D+00	.83264073D-04	-.79345307D+02

SIGMA	OMEGA	F(1,1)	F(1,2)
-.14140000D-01	.10000000D-02	-.94489818D+03	-.67119447D+01
-.14140000D-01	.20000000D-02	-.94490596D+03	-.68121800D+01
-.14140000D-01	.30000000D-02	-.94491893D+03	-.69792405D+01
-.14140000D-01	.40000000D-02	-.94493710D+03	-.72131290D+01
-.14140000D-01	.50000000D-02	-.94496049D+03	-.75138491D+01
-.14140000D-01	.60000000D-02	-.94498911D+03	-.78814056D+01
-.14140000D-01	.70000000D-02	-.94502298D+03	-.83158045D+01
-.14140000D-01	.80000000D-02	-.94506212D+03	-.88170527D+01
-.14140000D-01	.90000000D-02	-.94510658D+03	-.93851582D+01
-.14140000D-01	.10000000D-01	-.94515637D+03	-.10020130D+02
-.14140000D-01	.10000000D-01	-.94515637D+03	-.10020130D+02
-.14140000D-01	.20000000D-01	-.94595803D+03	-.20050329D+02
-.14140000D-01	.30000000D-01	-.94736210D+03	-.36785979D+02
-.14140000D-01	.40000000D-01	-.94948430D+03	-.60257407D+02
-.14140000D-01	.50000000D-01	-.95251921D+03	-.90512341D+02
-.14140000D-01	.60000000D-01	-.95678703D+03	-.12762277D+03
-.14140000D-01	.70000000D-01	-.96282398D+03	-.17169663D+03
-.14140000D-01	.80000000D-01	-.97156460D+03	-.22289795D+03
-.14140000D-01	.90000000D-01	-.98473545D+03	-.28148144D+03
-.14140000D-01	.10000000D+00	-.10057785D+04	-.34784919D+03
-.14140000D-01	.11000000D+00	-.10422290D+04	-.42261631D+03
-.14140000D-01	.12000000D+00	-.11123219D+04	-.50645639D+03
-.14140000D-01	.13000000D+00	-.12611497D+04	-.59769949D+03
-.14140000D-01	.14000000D+00	-.15308122D+04	-.67578849D+03
-.14140000D-01	.15000000D+00	-.15419810D+04	-.70842391D+03
-.14140000D-01	.16000000D+00	-.12411014D+04	-.78194962D+03
-.14140000D-01	.17000000D+00	-.10785195D+04	-.89405052D+03
-.14140000D-01	.18000000D+00	-.10102364D+04	-.10161699D+04
-.14140000D-01	.19000000D+00	-.97897700D+03	-.11435870D+04
-.14140000D-01	.20000000D+00	-.96327927D+03	-.12762809D+04
-.14140000D-01	.21000000D+00	-.95499120D+03	-.14146944D+04
-.14140000D-01	.22000000D+00	-.95068916D+03	-.15591987D+04
-.14140000D-01	.23000000D+00	-.94884061D+03	-.17100732D+04
-.14140000D-01	.24000000D+00	-.94882311D+03	-.18675438D+04
-.14140000D-01	.25000000D+00	-.95065669D+03	-.20318171D+04
-.14140000D-01	.26000000D+00	-.95524224D+03	-.22030825D+04
-.14140000D-01	.27000000D+00	-.96541849D+03	-.23812725D+04
-.14140000D-01	.28000000D+00	-.98593971D+03	-.25642092D+04
-.14140000D-01	.29000000D+00	-.99665066D+03	-.27461389D+04
-.14140000D-01	.30000000D+00	-.97585557D+03	-.29380006D+04
-.14140000D-01	.31000000D+00	-.96234285D+03	-.31427954D+04
-.14140000D-01	.32000000D+00	-.95739084D+03	-.33554111D+04
-.14140000D-01	.33000000D+00	-.95600604D+03	-.35748851D+04
-.14140000D-01	.34000000D+00	-.95616850D+03	-.38012392D+04
-.14140000D-01	.35000000D+00	-.95710395D+03	-.40345935D+04
-.14140000D-01	.36000000D+00	-.95848245D+03	-.42750556D+04
-.14140000D-01	.37000000D+00	-.96014735D+03	-.45227152D+04
-.14140000D-01	.38000000D+00	-.96201730D+03	-.47776499D+04
-.14140000D-01	.39000000D+00	-.96404686D+03	-.50399308D+04
-.14140000D-01	.40000000D+00	-.96620916D+03	-.53096248D+04
-.14140000D-01	.41000000D+00	-.96848765D+03	-.55867969D+04
-.14140000D-01	.42000000D+00	-.97087175D+03	-.58715117D+04
-.14140000D-01	.43000000D+00	-.97335460D+03	-.61638340D+04
-.14140000D-01	.44000000D+00	-.97593172D+03	-.64638293D+04
-.14140000D-01	.45000000D+00	-.97860019D+03	-.67715641D+04

PARTIAL PRODUCTION RUN FOR SIGMA = -.01414 PAGE 1

SIGMA	OMEGA	F(1,1)	F(1,2)
-.14140000D-01	.46000000D+00	-.98135818D+03	-.70871066D+04
-.14140000D-01	.47000000D+00	-.98420465D+03	-.74105266D+04
-.14140000D-01	.48000000D+00	-.98713911D+03	-.77418956D+04
-.14140000D-01	.49000000D+00	-.99016151D+03	-.80812873D+04
-.14140000D-01	.50000000D+00	-.99327212D+03	-.84287773D+04
-.14140000D-01	.51000000D+00	-.99647148D+03	-.87844437D+04
-.14140000D-01	.52000000D+00	-.99976033D+03	-.91483666D+04
-.14140000D-01	.53000000D+00	-.10031396D+04	-.95206290D+04
-.14140000D-01	.54000000D+00	-.10066104D+04	-.99013160D+04
-.14140000D-01	.55000000D+00	-.10101739D+04	-.10290516D+05
-.14140000D-01	.56000000D+00	-.10138315D+04	-.10688318D+05
-.14140000D-01	.57000000D+00	-.10175847D+04	-.11094818D+05
-.14140000D-01	.58000000D+00	-.10214348D+04	-.11510111D+05
-.14140000D-01	.59000000D+00	-.10253838D+04	-.11934296D+05
-.14140000D-01	.60000000D+00	-.10294331D+04	-.12367477D+05
-.14140000D-01	.61000000D+00	-.10335848D+04	-.12809759D+05
-.14140000D-01	.62000000D+00	-.10378407D+04	-.13261251D+05
-.14140000D-01	.63000000D+00	-.10422029D+04	-.13722066D+05
-.14140000D-01	.64000000D+00	-.10466735D+04	-.14192321D+05
-.14140000D-01	.65000000D+00	-.10512546D+04	-.14672135D+05
-.14140000D-01	.66000000D+00	-.10559486D+04	-.15161632D+05
-.14140000D-01	.67000000D+00	-.10607579D+04	-.15660941D+05
-.14140000D-01	.68000000D+00	-.10656849D+04	-.16170192D+05
-.14140000D-01	.69000000D+00	-.10707323D+04	-.16689523D+05
-.14140000D-01	.70000000D+00	-.10759027D+04	-.17219073D+05
-.14140000D-01	.71000000D+00	-.10811990D+04	-.17758989D+05
-.14140000D-01	.72000000D+00	-.10866241D+04	-.18309420D+05
-.14140000D-01	.73000000D+00	-.10921809D+04	-.18870522D+05
-.14140000D-01	.74000000D+00	-.10978727D+04	-.19442453D+05
-.14140000D-01	.75000000D+00	-.11037027D+04	-.20025380D+05
-.14140000D-01	.76000000D+00	-.11096744D+04	-.20619474D+05
-.14140000D-01	.77000000D+00	-.11157912D+04	-.21224911D+05
-.14140000D-01	.78000000D+00	-.11220569D+04	-.21841873D+05
-.14140000D-01	.79000000D+00	-.11284753D+04	-.22470549D+05
-.14140000D-01	.80000000D+00	-.11350503D+04	-.23111135D+05
-.14140000D-01	.81000000D+00	-.11417862D+04	-.23763833D+05
-.14140000D-01	.82000000D+00	-.11486873D+04	-.24428850D+05
-.14140000D-01	.83000000D+00	-.11557580D+04	-.25106404D+05
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5.3. EXTENSION TO DIGITAL SYSTEMS

The synthesis and analysis of the digital parameter plane is very similar in form to that of the continuous technique.[1] In order to understand it we will consider the third order system that is depicted in Fig. 5.6. The system can be normalized as indicated so that the results obtained will have a more general nature.

We are interested in the location of the roots relative to the unit circle (see Fig. 5.7.). Roots lying inside of the unit circle are stable, roots lying outside of the unit circle are unstable, and the unit circle then represents one of the stability boundaries. The other two boundaries, while actually lying on the unit circle have unique characteristics, are define at $z = \pm 1$.

We can now make the necessary substitution for z as we did for s and reduce the characteristic equation to two equations in two unknowns. The form then of the solution is basically the same whether one is talking about continuous domain or sampled domain systems (see Fig. 5.8.). Also in the digital systems one can make use of matrix formulations (Sec. 5.2.) and be able to accommodate high order systems with relative ease. An example of the digital domain parameter plane technique is contained in the following charts.

Fig. 5.6.

A SATELLITE DIGITAL CONTROLLER

CHARACTERISTIC EQUATION

$$\text{C. E.} = \sum_{j=0}^{3} \gamma_j z^j = 0, \qquad (6)$$

WHERE

$$\gamma_j = \underline{a}_j a + \underline{b}_j b + \underline{e}_j, \qquad (6J)$$

j	a_j	b_j	c_j
0	1	-1	2c -1
1	-2	0	2c +3
2	1	1	-3
3	0	0	1

MODIFIED GAINS a, b, c:

$$a \equiv K_D T/J_v, \qquad (7A)$$

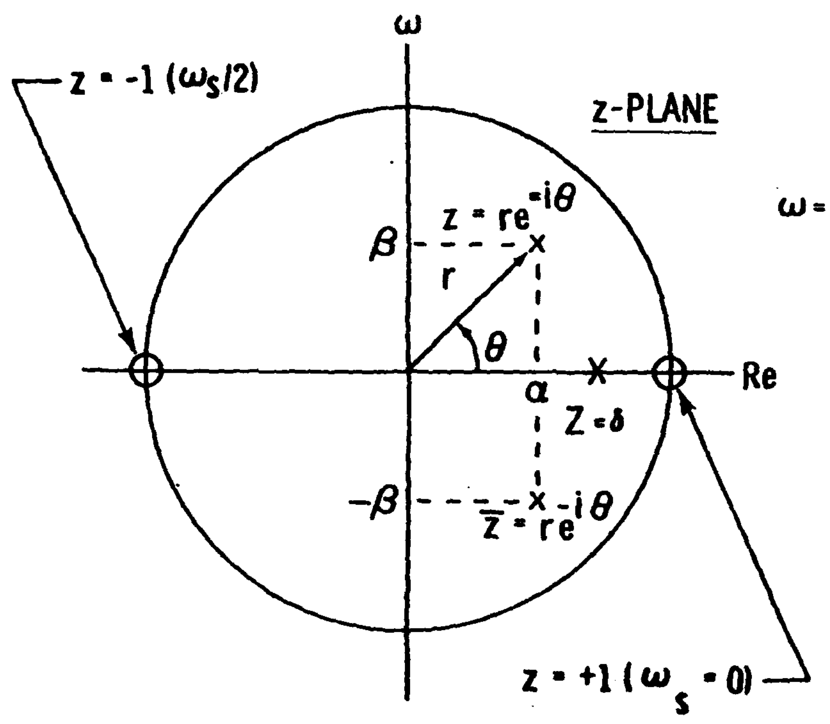
$$b \equiv K_P T^2/2J_v, \qquad (7B)$$

$$c \equiv K_I T^3/4J_v. \qquad (7C)$$

Fig. 5.7.

A SATELLITE DIGITAL CONTROLLER

UNIT CIRCLE IN z-PLANE



$$z = e^{Ts} = re^{i\theta}, \quad (8)$$

$$\theta = \omega T, \quad (9)$$

$$\omega = \omega_n (1-\zeta^2)^{1/2} \quad (10)$$

Fig. 5.8. COMPLEX CONJUGATE BOUNDARY AT $z = e^{i\theta}$

$$z^N = e^{iN\theta}$$

OR

$$z^N = \text{COS}\ (N\theta) + i\ \text{SIN}\ (N\theta)$$

$$CE = \sum_{j=0}^{N} (a_j a + b_j b + e_j)\ (\text{COS} j\theta + i\ \text{SIN} j\theta) = 0$$

DEFINE

$$A \equiv \sum_{j=0}^{N} a_j \text{COS} j\theta \ ; \ B \equiv \sum_{j=0}^{N} b_j \text{COS} j\theta$$

$$C \equiv \sum_{j=0}^{N} a_j \text{SIN} j\theta \ ; \ D \equiv \sum_{j=0}^{N} b_j \text{SIN} j\theta$$

$$E \equiv \sum_{j=0}^{N} e_j \text{COS} j\theta \ ; \ F \equiv \sum_{j=0}^{N} e_j \text{SIN} j\theta$$

THUS

$$\begin{aligned} aA + bB &= E \\ aC + bD &= F \end{aligned} \quad \Rightarrow \quad a \ \& \ b \quad (A,B,C,D,E,F)$$

5.4 EXAMPLE

In order to portray the power of that parameter plane technique we have chosen the model as depicted on Fig. 5.9. Here we have a general digital control system (PD) as applied to the end of a free-free structure. The controller utilizes a rate signal in the feedback loop that is digitally integrated to yield an estimate of the vehicle attitude. The parameter plane for this configuration is given on the following pages for one and then three bending modes in the model.

The parameter plane for one bending mode is presented in Fig. 5.10. The stability boundaries and the constant damping contours are plotted as function of the two control gains, position (a_0) and rate (a_1). The stable region is indicated by the cross-hatched region. Also presented are contours of constant closed loop damping. The designer can then select the values of the two control gains to achieve the stability and response that he desires. Also, one would be well advised to select a set of gains that are comfortably within the stability region (robustness). The plot on Fig. 5.11 is a "blow-up" of the origin of the stability region to illustrate the movement of the constant damping contours in the vicinity of the origin. It is important that the control system analyst carefully investigate all regions of the parameter plane to insure that no "funnies" go unnoticed. This fact will become more apparent as we investigate the stability boundaries and contours for the same control

configuration with the addition of the next two bending modes. This information is contained on Fig. 5.12 - 5.13.

The parameter plane including three bending modes is presented on Fig. 5.12. The scale utilized is the same scale as for the one bending mode case to emphasize the reduction in the stability region. This graph forecefully depicts the requirement for adequate bending suppression in high order systems. The plot on Fig. 5.13 is a more reasonable scaling of the stable region for the three bending mode case. The curves in this figure can be used as guides in selection of the appropriate gains to use in not only insuring stability but also in providing the requisite response characteristics.

Again a "blow-up" of the region in the vicinity of the origin is shown in Fig. 5.14. The motion of the constant damping contours becomes more interesting as the order of the system is increased. Also we must be very careful of the increment that is used to vary $w_n t$ along the constant damping contour.

Fig. 5.9 Simplified Block Diagram

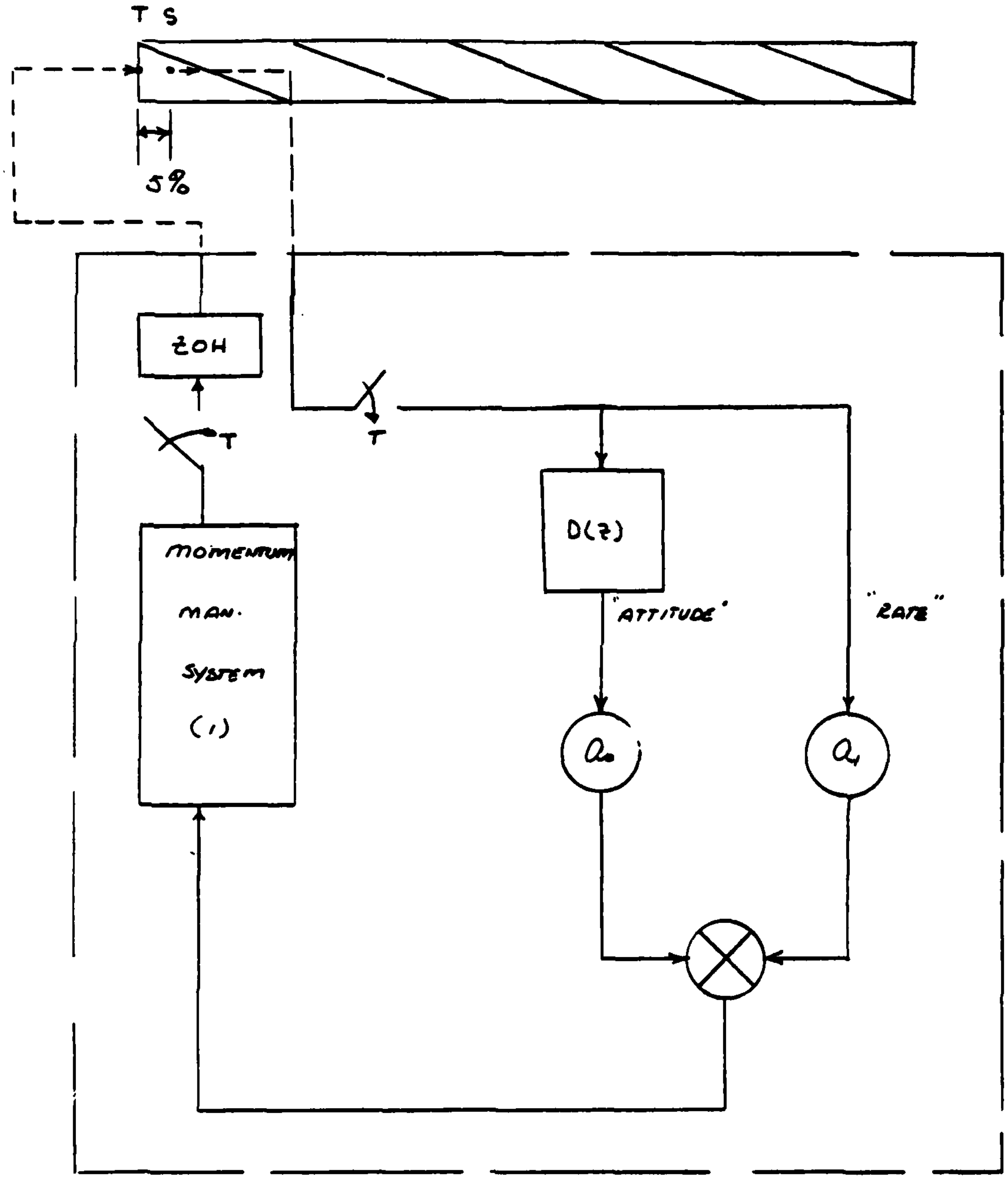
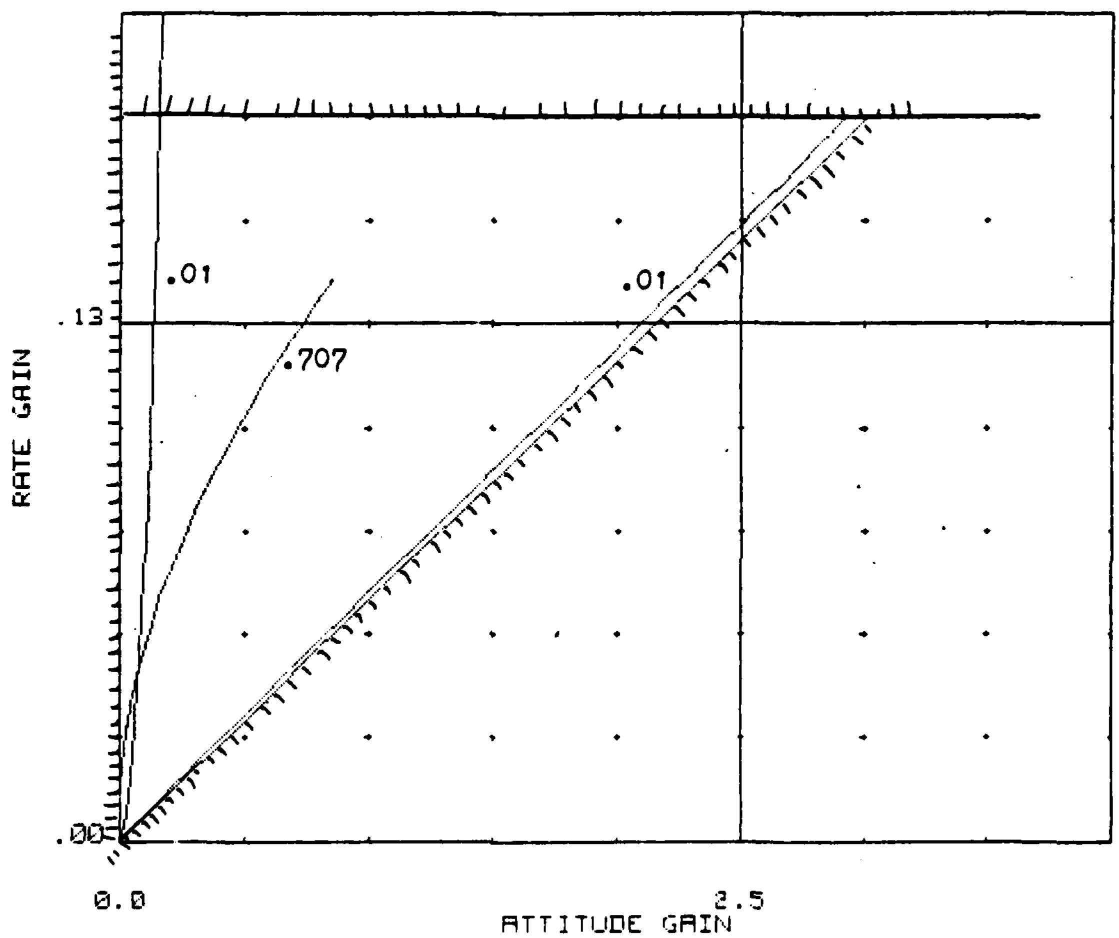


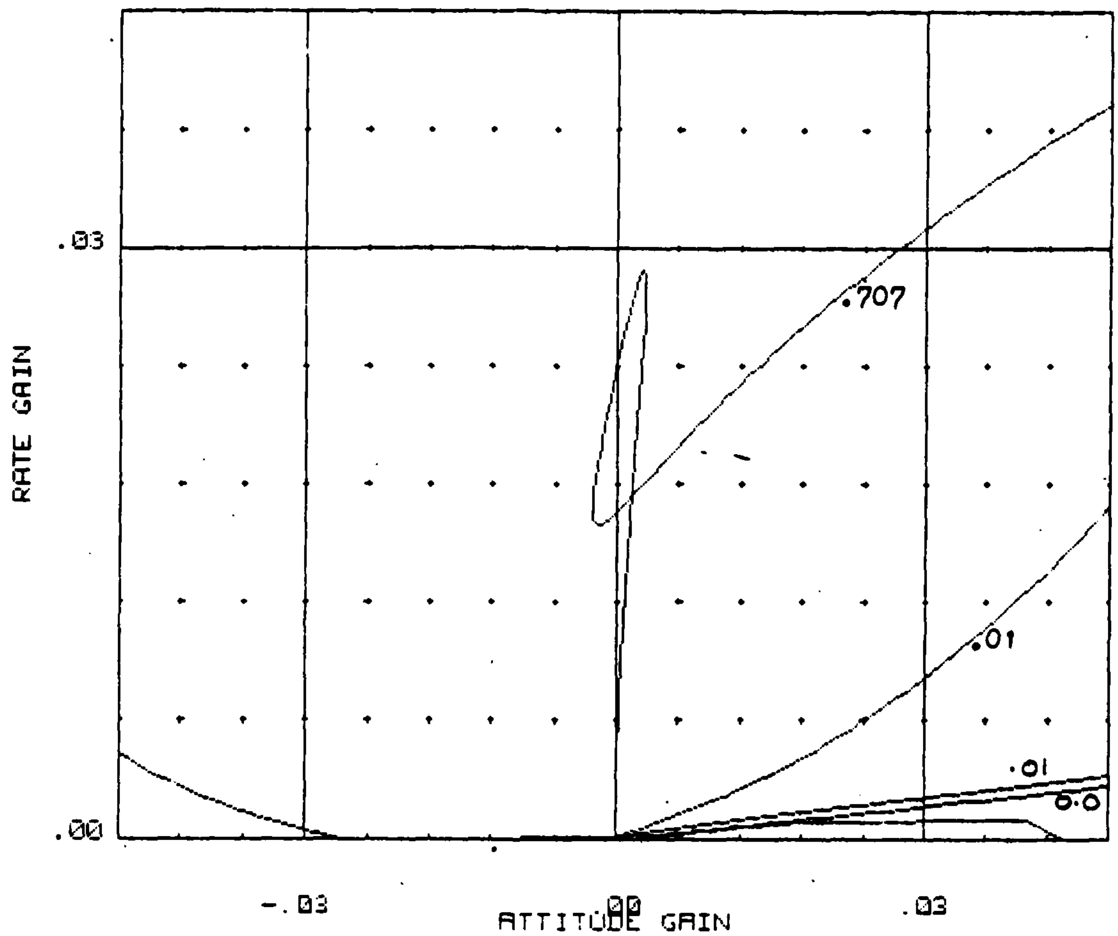
Fig. 5.10 One bending mode parameter plane



w_b 1.00 rad/sec

T 0.11626 sec

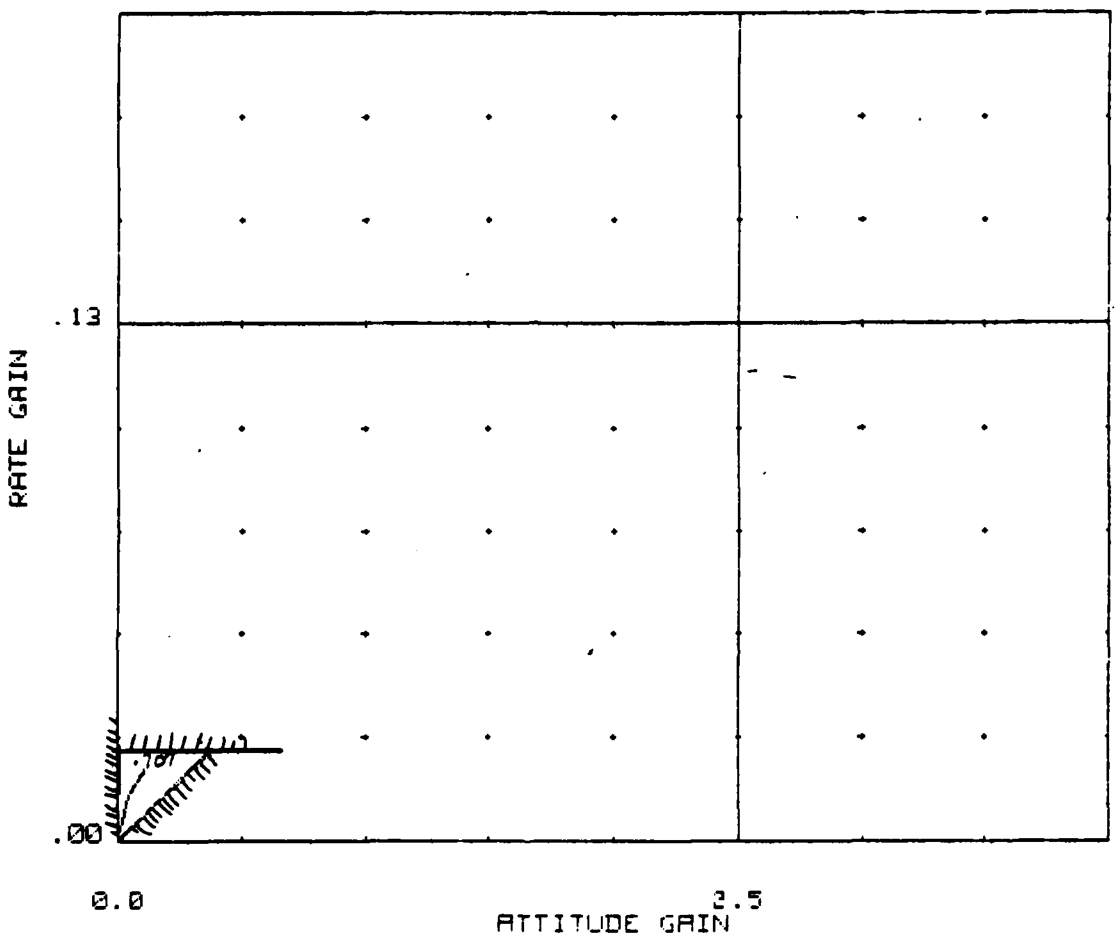
Fig. 5.11   One bending mode parameter plane



w_b 1.00 rad/sec

T   0.1162 seconds

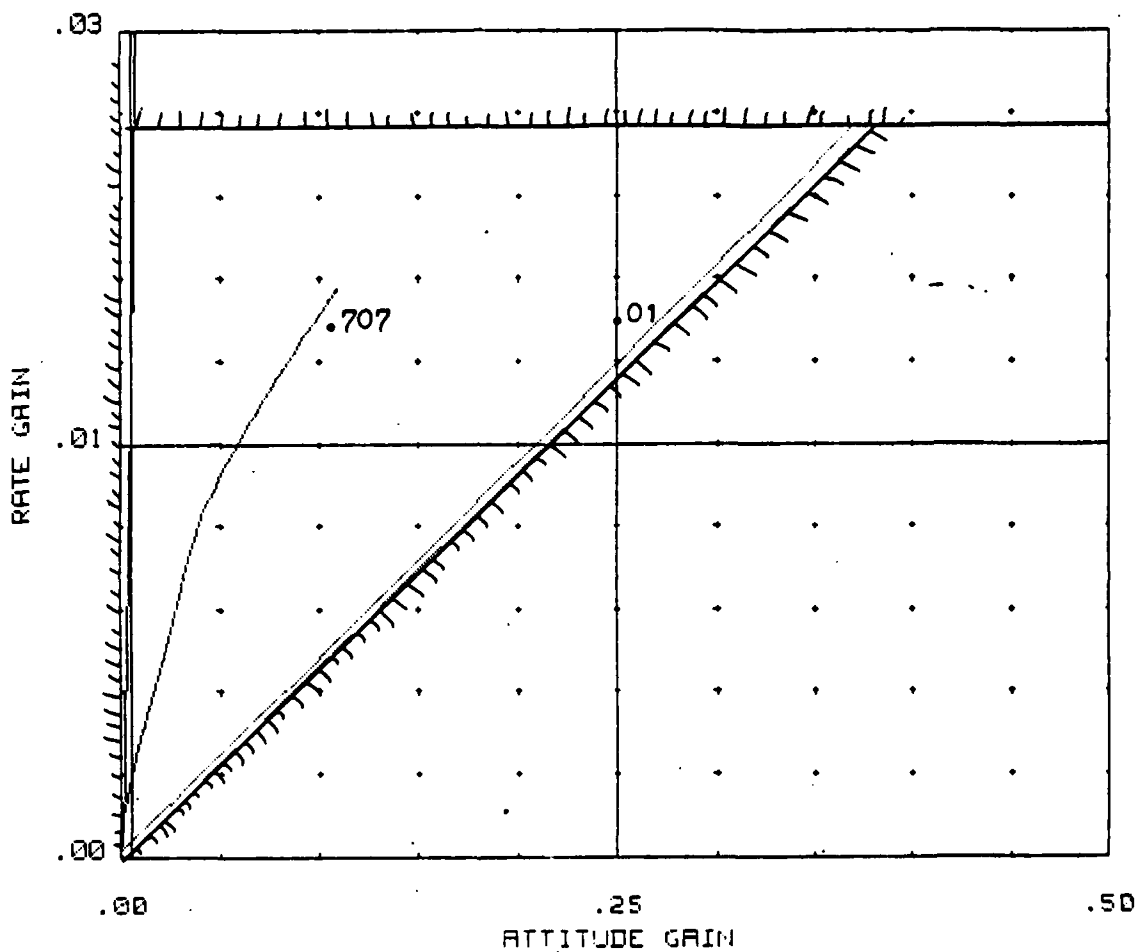
Three bending modes parameter plane   Fig. 5.12



w_b 1.00,2.76,5.40 rad/sec

T   0.11626 seconds (10 samples of 5.40 component)

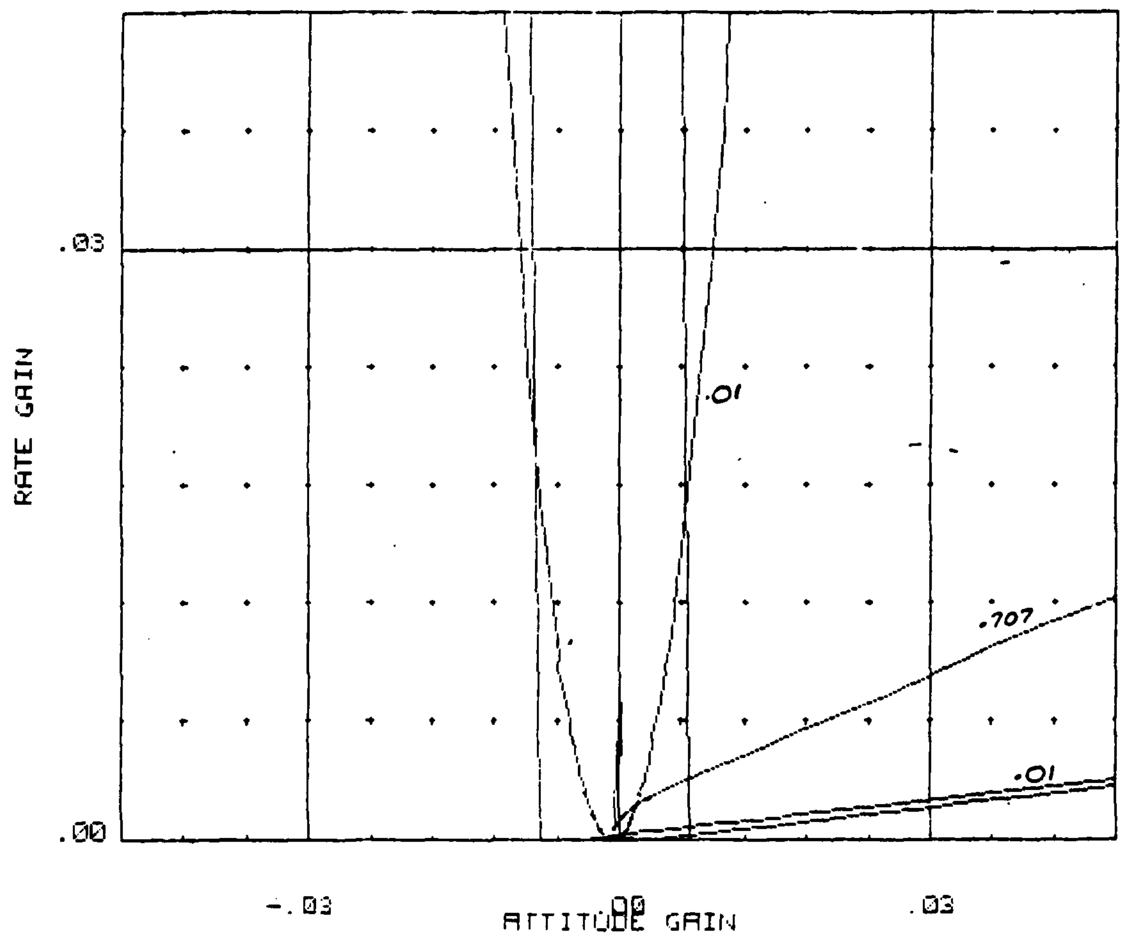
Three bending mode parameter plane   5.13



w_b 1.00,2.76,5.40 rad/sec

T   0.11626 seconds

Three bending mode parameter plane   Fig. 5.14



w_b 1.00,2.76,5.40 rad/sec

T 0.11626 seconds

5.5 SUMMARY

In summary, we have presented a technique, valid for both continuous and digital control system design, that is both powerful in concept and application but very simple to understand and to use. We have applied this theory to a number of vehicles and configurations over the years and have found it to be extremely effective. The recent advancements under this contract to extend this technique to include high order systems has opened the door to exciting and interesting future applications in the analysis and design of future large space structures. However, further effort is indicated to handle cases whose characteristic equations have closely-spaced roots.

5.6 PEFERENCES

[1] Seltzer, S.M., "Application of the Parameter Space Method to Aerospace Vehicle Digital Control System Design," IEEE Transactions on Automatic Control, Vol. AC-26, No.2, April 1981, pp. 530-533.

6.0 CONCLUSIONS

This report has presented an overview of the activities of the Control Dynamics Company in performing the requirements of the subject contract. We have presented a study of active structural control technology as applied to Large Space Structures (LSS). In particular, we have presented the development of analytical tools for analyzing the behavior of LSS's and presented the development of means to ensure LSS stability and performance. In addition we have supported the overall ACOSS activities to develop and understand a generic, unified, structural dynamics and control technology base for LSS. To this end we have presented an assessment of both ACOSS modeling activities and control system analysis activities.

We have shown that simple analysis tools can be utilized to advantage in the LSS arena. Through the application of simple and understandable tools, a "high order" structural model was developed and presented herein. Also "simple" analysis tools were presented for analysis of control system stability and performance characteristics, both in the continuous and sampled domains. The insight and information gained through application of these techniques was employed in participation with all ACOSS Contractors throughout the term of this contract.

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